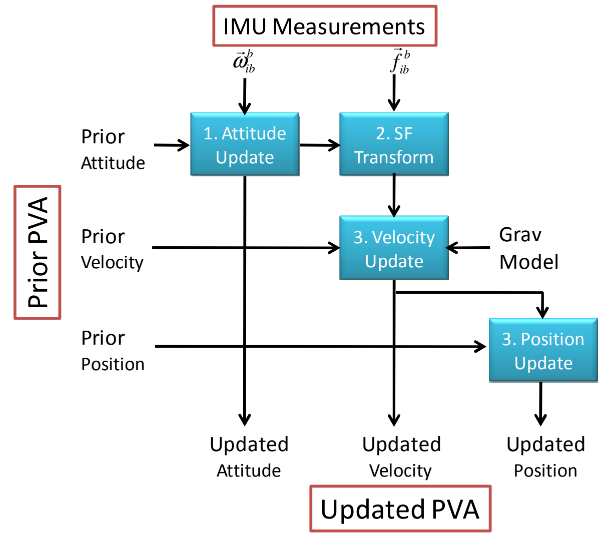
INS Initialization



- How can we initialize the INS?
  - Position and Velocity
    - Typically from GPS or other position fixing source
  - Attitude (Self-Alignment)
    - Can be determined by a suitably accurate IMU if stationary
      - For an ECI mechanization a star tracker is a common alternative
    - Effectively, the IMU "sees" the earth rate vector and g-vector (i.e. acceleration due to gravity)
      - This will not work for low-grade inertial sensors
        - » The gyros must have biases much less than earth rate

INS Initialization

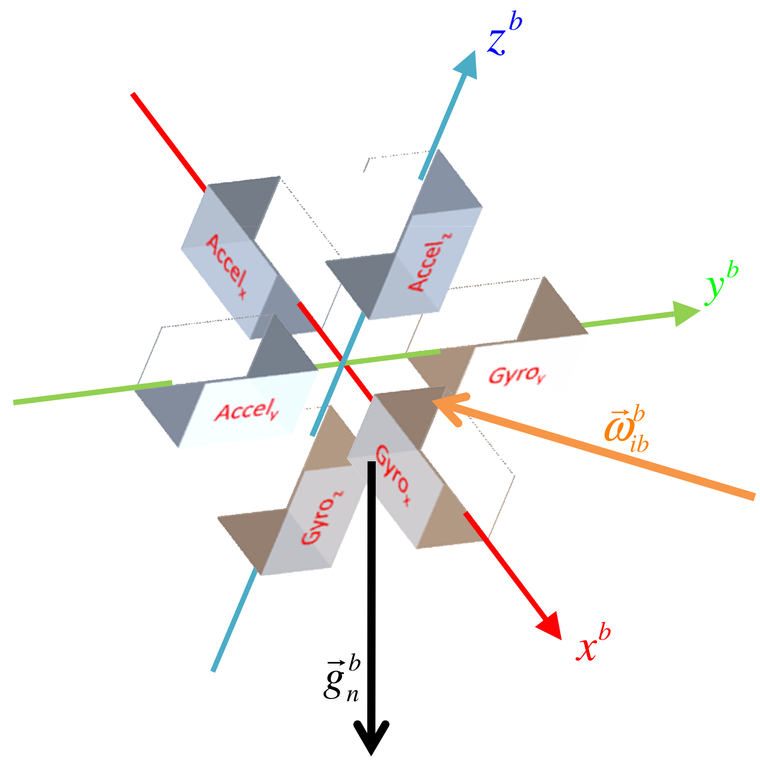
Attitude Initialization (Self-Alignment)

- The three gyros "see" earth rate

$$\vec{\omega}_{ib}^{b} = C_n^b C_e^n \vec{\omega}_{ie}^{e} = C_n^b \vec{\omega}_{ie}^{n} \simeq C_n^b \begin{bmatrix} \omega_{ie}\cos(L_b) \\ 0 \\ -\omega_{ie}\sin(L_b) \end{bmatrix}$$

- The three accels "see" the g-vector
  - g is a function of lat & lon

$$\vec{f}_{ib}^{\,b} = C_n^b \vec{g}_b^{\,n} \simeq C_n^b \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$



INS Initialization

Attitude Initialization (Self-Alignment)

- Can generate a third vector equation

$$\vec{f}^{\,b}_{ib} \times \vec{\omega}^{b}_{ib} \simeq C^{b}_{n} \left(\begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \times \begin{bmatrix} \omega_{ie} \cos(L_b) \\ 0 \\ -\omega_{ie} \sin(L_b) \end{bmatrix} \right)$$

- Using measurements of the specific force ($\tilde{\vec{f}}^{\,b}_{ib}$), angular velocity ($\tilde{\vec{\omega}}^{b}_{ib}$), and cross-product ($\tilde{\vec{f}}^{\,b}_{ib} \times \tilde{\vec{\omega}}^{b}_{ib}$)

$$\begin{bmatrix} \tilde{\vec{f}}^{\,b}_{ib} & \tilde{\vec{\omega}}^{b}_{ib} & \left(\tilde{\vec{f}}^{\,b}_{ib} \times \tilde{\vec{\omega}}^{b}_{ib} \right) \end{bmatrix} = C^{b}_{n} \begin{bmatrix} \vec{g}^{\,n}_{b} & \vec{\omega}^{n}_{ie} & \left(\vec{g}^{\,n}_{b} \times \vec{\omega}^{n}_{ie} \right) \end{bmatrix}$$

$$[A] = C^{b}_{n} [B] \Rightarrow [A][B]^{-1} = C^{b}_{n} \Rightarrow C^{n}_{b} = \left([A][B]^{-1} \right)^{T} = [B]^{-T} [A]^{T}$$

INS Initialization

Attitude Initialization (Self-Alignment)

- With some help from Mathematica™
  - This is often referred to as course-alignment
    - The DCM can now be reduced to Euler angles, a quaternion, or other suitable orientation representations

$$C_b^n = \begin{bmatrix} \frac{\tan(L_b)}{g} & \frac{1}{\omega_{ie}\cos(L_b)} & 0 \\ 0 & 0 & \frac{1}{g\,\omega_{ie}\cos(L_b)} \\ \frac{1}{g} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\vec{f}}_{ib}^{\,b} & \tilde{\vec{\omega}}_{ib}^{\,b} & \left(\tilde{\vec{f}}_{ib}^{\,b} \times \tilde{\vec{\omega}}_{ib}^{\,b}\right) \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{\omega_x}{\omega_{ie}\cos(\theta_{\text{lat}})} + \frac{f_x \tan(L_b)}{g} & \frac{\omega_y}{\omega_{ie}\cos(\theta_{\text{lat}})} + \frac{f_y \tan(L_b)}{g} & \frac{\omega_z}{\omega_{ie}\cos(\theta_{\text{lat}})} + \frac{f_z \tan(L_b)}{g} \\ \frac{\left(-f_z\omega_y + f_y\omega_z\right)}{g\,\omega_{ie}\cos(L_b)} & \frac{\left(f_z\omega_x - f_x\omega_z\right)}{g\,\omega_{ie}\cos(L_b)} & \frac{\left(-f_y\omega_x + f_x\omega_y\right)}{g\,\omega_{ie}\cos(L_b)} \\ \frac{f_x}{g} & \frac{f_y}{g} & \frac{f_z}{g} \end{bmatrix}$$

INS Error Propagation

- Focus on errors due to deterministic error sources
  - Sensor errors
    - Biases, misalignment, scale factor, …
    - Noise contributors (ARW, VRW, …) are also significant
  - Initialization errors
    - Imperfect initial position, velocity and attitude
  - Mechanization equations introduce errors
    - Can increase fidelity and/or update rates
    - Gravity model

INS Error Propagation

- Define orientation error as
  - Measured × Truth^{-1} (DCM)

$$\delta C_\beta^\alpha = \tilde{C}_\beta^\alpha \; C_\alpha^\beta \;\; = e^{\Omega_{\alpha\beta}^\alpha dt} = e^{\left[\vec{\varpi}_{\alpha\beta}^\alpha \times\right] dt} = e^{\left[\delta\vec{\psi}_{\alpha\beta}^\alpha \times\right]} = I + \left[\delta\vec{\psi}_{\alpha\beta}^\alpha \times\right] + \ldots$$

$$\therefore \;\; \left[\delta\vec{\psi}_{\alpha\beta}^\alpha \times\right] \simeq \delta C_\beta^\alpha - I \qquad\qquad \Rightarrow \;\; \left[\delta\dot{\vec{\psi}}_{\alpha\beta}^\alpha \times\right] \simeq \delta\dot{C}_\beta^\alpha$$

- Define linear (velocity, pos, …) errors as
  - Measured - Truth

$$\delta\vec{v}_{\beta\alpha}^\gamma = \tilde{\vec{v}}_{\beta\alpha}^\gamma - \vec{v}_{\beta\alpha}^\gamma$$

$$\delta\vec{\omega}_{ib}^b = \tilde{\vec{\omega}}_{ib}^b - \vec{\omega}_{ib}^b \simeq b_g$$

$$\delta\vec{f}_{ib}^b = \tilde{\vec{f}}_{ib}^b - \vec{f}_{ib}^b \simeq b_a$$

INS Error Propagation

- Want to build a state-space error model
  - State vector

$$\vec{x} = \begin{bmatrix} \delta(PVA) \\ Gyro / Accel\ Biases \end{bmatrix} = \begin{bmatrix} \delta\vec{\psi} \\ \delta\vec{v} \\ \delta\vec{r} \\ \vec{b}_a \\ \vec{b}_g \end{bmatrix}$$

  - This will allow us to model the system's error dynamics

$$\dot{\vec{x}} = A\vec{x} = A \begin{bmatrix} \delta\vec{\psi} \\ \delta\vec{v} \\ \delta\vec{r} \\ \vec{b}_a \\ \vec{b}_g \end{bmatrix}$$

$$\boxed{\dot{\vec{b}}_a = \dot{\vec{b}}_g = \vec{0}}$$

INS Error Propagation

Case I: ECI Error Model

$$\vec{x} = \begin{bmatrix} \delta\vec{\psi}_{ib}^i \\ \delta\vec{v}_{ib}^i \\ \delta\vec{r}_{ib}^i \\ \vec{b}_a \\ \vec{b}_g \end{bmatrix}$$

- Error model state vector
  - Attitude Error
    - Recall that: $\dot{C}_b^i = C_b^i \; \Omega_{ib}^b$

$$\delta C_b^i = \tilde{C}_b^i \; C_i^b \quad \Rightarrow \delta \dot{C}_b^i = \dot{\tilde{C}}_b^i \; C_i^b + \tilde{C}_b^i \; \dot{C}_i^b$$

$$\delta \dot{C}_b^i \simeq \left[\delta\dot{\vec{\psi}}_{ib}^i \times \right]$$

- Thus

$$\dot{C}_i^b = C_i^b \; \Omega_{bi}^i$$

$$\dot{\tilde{C}}_b^i = \tilde{C}_b^i \; \tilde{\Omega}_{ib}^b$$

$$\left[(C\omega)\times\right] = C\left[\omega\times\right]C^T$$

$$\Rightarrow \Omega^b = C_i^b \Omega^i C_b^i$$

$$\left[\delta\dot{\vec{\psi}}_{ib}^i \times \right] = \dot{\tilde{C}}_b^i \; C_i^b + \tilde{C}_b^i \; \dot{C}_i^b$$

$$= \tilde{C}_b^i \; \tilde{\Omega}_{ib}^b \; C_i^b + \tilde{C}_b^i \; C_i^b \; \Omega_{bi}^i$$

$$= \tilde{C}_b^i \left[\tilde{\Omega}_{ib}^b \; C_i^b - C_i^b \; \Omega_{ib}^i \right]$$

$$= \tilde{C}_b^i \left[\tilde{\Omega}_{ib}^b - \Omega_{ib}^b \right] C_i^b$$

$$= \tilde{C}_b^i \left[\vec{b}_g \times \right] C_i^b \simeq \left[C_b^i \vec{b}_g \times \right]$$

$$\therefore \quad \delta\dot{\vec{\psi}}_{ib}^i \simeq C_b^i \vec{b}_g$$

INS Error Propagation

Case I: ECI Error Model

- Velocity Error
  - Recall that: $\vec{a}_{ib}^{i} = \dot{\vec{v}}_{ib}^{i} = \vec{f}_{ib}^{i} - \vec{\gamma}_{ib}^{i} = C_b^i \vec{f}_{ib}^{b} - \vec{\gamma}_{ib}^{i}$

$$\delta\dot{\vec{v}}_{ib}^{i} = \dot{\tilde{\vec{v}}}_{ib}^{i} - \dot{\vec{v}}_{ib}^{i}$$

$$= \tilde{C}_b^i \tilde{\vec{f}}_{ib}^{b} - \tilde{\vec{\gamma}}_{ib}^{i} - \left(C_b^i \vec{f}_{ib}^{b} - \vec{\gamma}_{ib}^{i}\right) = \tilde{C}_b^i \tilde{\vec{f}}_{ib}^{b} - C_b^i \vec{f}_{ib}^{b} - \left[\tilde{\vec{\gamma}}_{ib}^{i} - \vec{\gamma}_{ib}^{i}\right]$$

  - Now, by adding zero

$$\tilde{C}_b^i \tilde{\vec{f}}_{ib}^{b} - C_b^i \vec{f}_{ib}^{b} + \left(C_b^i \tilde{\vec{f}}_{ib}^{b} - C_b^i \tilde{\vec{f}}_{ib}^{b}\right) = \left[\tilde{C}_b^i - C_b^i\right]\tilde{\vec{f}}_{ib}^{b} + C_b^i\left[\tilde{\vec{f}}_{ib}^{b} - \vec{f}_{ib}^{b}\right]$$

$$= \left[\delta C_b^i C_b^i - C_b^i\right]\left(\vec{f}_{ib}^{b} + b_a\right) + C_b^i\left[b_a\right]$$

$$\simeq \left[\delta C_b^i - I\right] C_b^i \vec{f}_{ib}^{b} + C_b^i b_a$$

$$\simeq \delta\vec{\psi}_{ib}^{i} \times C_b^i \vec{f}_{ib}^{b} + C_b^i b_a$$

$$= -\left[C_b^i \vec{f}_{ib}^{b} \times\right] \delta\vec{\psi}_{ib}^{i} + C_b^i b_a$$

$$\delta C_b^i = \tilde{C}_b^i \, C_i^b$$

$$\delta C_b^i \simeq I + \left[\delta\vec{\psi}_{ib}^{i} \times\right]$$

INS Error Propagation

Case I: ECI Error Model

- The remaining term
  - Recall that:

$$\vec{\gamma}_{ib}^{i} \approx \frac{\left(r_{eS}^{e}(L_b)\right)^2}{\left(r_{eS}^{e}(L_b)+h_b\right)^2}\vec{\gamma}_0^{i}(L_b)$$

- Now, by ignoring gravitational variation w/ latitude and making additional simplifications

$$\left[\tilde{\vec{\gamma}}_{ib}^{i} - \vec{\gamma}_{ib}^{i}\right] \simeq \frac{2g_0}{r_{eS}^{e}}\left[\vec{r}_{ib}^{\,i}\left(\vec{r}_{ib}^{\,i}\right)^T\right]\frac{\delta\vec{r}_{ib}^{\,i}}{\left\|\vec{r}_{ib}^{\,i}\right\|}$$

- We finally obtain

$$\delta\dot{\vec{v}}_{ib}^{i} = -\left[C_b^i \vec{f}_{ib}^{\,b}\times\right]\delta\vec{\psi}_{ib}^{i} + C_b^i b_a + \frac{2g_0}{\left\|\vec{r}_{ib}^{\,i}\right\| r_{eS}^{e}}\left[\vec{r}_{ib}^{\,i}\left(\vec{r}_{ib}^{\,i}\right)^T\right]\delta\vec{r}_{ib}^{\,i}$$

INS Error Propagation

Case I: ECI Error Model

- Position Error
  - Recalling that $\dot{\vec{r}}_{ib}^{i} = \vec{v}_{ib}^{i}$ $\Rightarrow \delta\dot{\vec{r}}_{ib}^{i} = \delta\vec{v}_{ib}^{i}$

- In Summary:

$$\begin{bmatrix} \delta\dot{\vec{\psi}}_{ib}^{i} \\ \delta\dot{\vec{v}}_{ib}^{i} \\ \delta\dot{\vec{r}}_{ib}^{i} \\ \dot{\vec{b}}_{a} \\ \dot{\vec{b}}_{g} \end{bmatrix} = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & C_b^i \\ -\left[C_b^i \vec{f}_{ib}^{\,b} \times\right] & 0_{3\times3} & \frac{2g_0}{\left\|\vec{r}_{ib}^{i}\right\| r_{eS}^{e}}\left[\vec{r}_{ib}^{i}\left(\vec{r}_{ib}^{i}\right)^T\right] & C_b^i & 0_{3\times3} \\ 0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ & & 0_{3\times15} & & \\ & & 0_{3\times15} & & \end{bmatrix} \begin{bmatrix} \delta\vec{\psi}_{ib}^{i} \\ \delta\vec{v}_{ib}^{i} \\ \delta\vec{r}_{ib}^{i} \\ \vec{b}_{a} \\ \vec{b}_{g} \end{bmatrix}$$

- This continuous time model can be converted to a discrete model

INS Error Propagation

Case II: ECEF Error Model

$$\vec{x} = \begin{bmatrix} \delta\vec{\psi}^e_{eb} \\ \delta\vec{v}^e_{eb} \\ \delta\vec{r}^e_{eb} \\ \vec{b}_a \\ \vec{b}_g \end{bmatrix}$$

- Error model state vector
  - Attitude Error
    - Recall that: $\dot{C}^e_b = C^e_b \, \Omega^e_{eb}$
    - Leads again to

$$\left[\delta\dot{\vec{\psi}}^e_{eb} \times\right] = \tilde{C}^e_b \left[\tilde{\Omega}^b_{eb} - \Omega^b_{eb}\right] C^b_e$$

$$\vec{\omega}^b_{ib} = \vec{\omega}^b_{ie} + \vec{\omega}^b_{eb}$$

$$\vec{\omega}^b_{eb} = \vec{\omega}^b_{ib} - \vec{\omega}^b_{ie} = \vec{\omega}^b_{ib} - C^b_e \vec{\omega}^e_{ie}$$

    - Thus

$$\begin{aligned} \delta\dot{\vec{\psi}}^e_{eb} &\simeq C^e_b \left[\tilde{\vec{\omega}}^b_{eb} - \vec{\omega}^b_{eb}\right] \\ &= C^e_b \left[\left(\tilde{\vec{\omega}}^b_{ib} - \tilde{C}^b_e \tilde{\vec{\omega}}^e_{ie}\right) - \left(\vec{\omega}^b_{ib} - C^b_e \vec{\omega}^e_{ie}\right)\right] \\ &= C^e_b \left(\tilde{\vec{\omega}}^b_{ib} - \vec{\omega}^b_{ib}\right) + \left(I - C^e_b \tilde{C}^b_e\right) \vec{\omega}^e_{ie} \\ &\simeq C^e_b \vec{b}_g + \left[\delta\vec{\psi}^e_{eb} \times\right] \vec{\omega}^e_{ie} \end{aligned}$$

$$\therefore \quad \delta\dot{\vec{\psi}}^e_{eb} \simeq C^e_b \vec{b}_g - \left[\vec{\omega}^e_{ie} \times\right] \delta\vec{\psi}^e_{eb}$$

INS Error Propagation

Case II: ECEF Error Model

- Velocity Error
  - Recalling that $\dot{\vec{v}}^e_{eb} = \vec{f}^e_{ib} + \vec{g}^e_b - 2\Omega^i_{ie}\vec{v}^e_{eb}$

$$= \vec{f}^e_{ib} - 2\Omega^i_{ie}\vec{v}^e_{eb} + \gamma^e_{ib} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^e_{eb}$$

  - Following the prior analysis

$$\delta\dot{\vec{v}}^e_{eb} = \dot{\tilde{\vec{v}}}^e_{eb} - \dot{\vec{v}}^e_{eb}$$

$$= \left(\tilde{\vec{f}}^e_{ib} - \vec{f}^e_{ib}\right) - 2\Omega^i_{ie}\left(\tilde{\vec{v}}^e_{eb} - \vec{v}^e_{eb}\right) + \left(\tilde{\vec{\gamma}}^e_{ib} - \vec{\gamma}^e_{ib}\right) - \Omega^e_{ie}\Omega^e_{ie}\left(\tilde{\vec{r}}^e_{eb} - \vec{r}^e_{eb}\right)$$

$$\left(\tilde{\vec{f}}^e_{ib} - \vec{f}^e_{ib}\right) \rightarrow -\left[C^e_b\vec{f}^b_{ib}\times\right]\delta\vec{\psi}^e_{eb} + C^e_b b_a$$

$$2\Omega^i_{ie}\left(\tilde{\vec{v}}^e_{eb} - \vec{v}^e_{eb}\right) \rightarrow 2\Omega^i_{ie}\delta\vec{v}^e_{eb}$$

$$\left(\tilde{\vec{\gamma}}^e_{ib} - \vec{\gamma}^e_{ib}\right) \rightarrow \frac{2g_0}{r^e_{eS}}\left[\vec{r}^e_{eb}\left(\vec{r}^e_{eb}\right)^T\right]\frac{\delta\vec{r}^e_{eb}}{\left\|\vec{r}^e_{eb}\right\|}$$

$$\Omega^e_{ie}\Omega^e_{ie}\left(\tilde{\vec{r}}^e_{eb} - \vec{r}^e_{eb}\right) \rightarrow \Omega^e_{ie}\Omega^e_{ie}\delta\vec{r}^e_{eb} \approx \vec{0}$$

$$\delta\dot{\vec{v}}^e_{eb} = -\left[C^e_b\vec{f}^b_{ib}\times\right]\delta\vec{\psi}^e_{eb} + C^e_b b_a - 2\Omega^i_{ie}\delta\vec{v}^e_{eb} + \frac{2g_0}{\left\|\vec{r}^e_{eb}\right\| r^e_{eS}}\left[\vec{r}^e_{eb}\left(\vec{r}^e_{eb}\right)^T\right]\delta\vec{r}^e_{eb}$$

INS Error Propagation

Case II: ECEF Error Model

- Position Error
  - Recalling that $\dot{\vec{r}}^{e}_{eb} = \vec{v}^{e}_{eb}$ $\Rightarrow \delta\dot{\vec{r}}^{e}_{eb} = \delta\vec{v}^{e}_{eb}$

- In Summary:

$$\begin{bmatrix} \delta\dot{\vec{\psi}}^{e}_{eb} \\ \delta\dot{\vec{v}}^{e}_{eb} \\ \delta\dot{\vec{r}}^{e}_{eb} \\ \dot{\vec{b}}_a \\ \dot{\vec{b}}_g \end{bmatrix} = \begin{bmatrix} -\left[\vec{\omega}^{e}_{ie}\times\right] & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & C^{e}_{b} \\ -\left[C^{e}_{b}\vec{f}^{b}_{ib}\times\right] & -2\Omega^{i}_{ie} & \frac{2g_0}{\left\|\vec{r}^{e}_{eb}\right\| r^{e}_{eS}}\left[\vec{r}^{e}_{eb}\left(\vec{r}^{e}_{eb}\right)^{T}\right]\delta\vec{r}^{e}_{eb} & C^{e}_{b} & \phi_{3\times3} \\ 0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ & & 0_{3\times15} & & \\ & & 0_{3\times15} & & \end{bmatrix} \begin{bmatrix} \delta\vec{\psi}^{e}_{eb} \\ \delta\vec{v}^{e}_{eb} \\ \delta\vec{r}^{e}_{eb} \\ \vec{b}_a \\ \vec{b}_g \end{bmatrix}$$

- This continuous time model can be converted to a discrete model