EE 570: Location and Navigation

On-Line Bayesian Tracking

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Objective

Sequentially estimate on-line the states of a system as it changes over time using observations that are corrupted with noise.

Given State-Space Equations

$$\vec{\boldsymbol{x}}_k = \mathbf{f}_k(\vec{\boldsymbol{x}}_{k-1}, \vec{\boldsymbol{w}}_{k-1}) \tag{1}$$

$$\vec{\boldsymbol{z}}_k = \mathbf{h}_k(\vec{\boldsymbol{x}}_k, \vec{\boldsymbol{v}}_k) \tag{2}$$

Given State-Space Equations

$(n \times 1)$ state vector at time k

$$\vec{x}_k = \mathbf{f}_k(\vec{x}_{k-1}, \vec{w}_{k-1}) \quad (1)$$

$$\vec{z}_k = \mathbf{h}_k(\vec{x}_k, \vec{v}_k) \quad (2)$$

$(m \times 1)$ measurement vector at time k

Given State-Space Equations

Possibly non-linear function,
$\mathbf{f}_k : \Re^n \times \Re^{n_w} \mapsto \Re^n$

$$\vec{x}_k = \mathbf{f}_k(\vec{x}_{k-1}, \vec{w}_{k-1}) \quad (1)$$

$$\vec{z}_k = \mathbf{h}_k(\vec{x}_k, \vec{v}_k) \quad (2)$$

Possibly non-linear function,
$\mathbf{h}_k : \Re^m \times \Re^{n_v} \mapsto \Re^m$

Given State-Space Equations

i.i.d state noise

$$\vec{\boldsymbol{x}}_k = \mathbf{f}_k(\vec{\boldsymbol{x}}_{k-1}, \vec{\boldsymbol{w}}_{k-1}) \quad (1)$$

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i.i.d measurement noise

Given State-Space Equations

$$\vec{\boldsymbol{x}}_k = \mathbf{f}_k(\vec{\boldsymbol{x}}_{k-1}, \vec{\boldsymbol{w}}_{k-1}) \quad (1)$$

$$\vec{\boldsymbol{z}}_k = \mathbf{h}_k(\vec{\boldsymbol{x}}_k, \vec{\boldsymbol{v}}_k) \quad (2)$$

The state process is Markov chain, i.e., $p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{x}}_1, \ldots, \vec{\boldsymbol{x}}_{k-1}) = p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{x}}_{k-1})$ and the distribution of $\vec{\boldsymbol{z}}_k$ conditional on the state $\vec{\boldsymbol{x}}_k$ is independent of previous state and measurement values, i.e., $p(\vec{\boldsymbol{z}}_k|\vec{\boldsymbol{x}}_{1:k}, \vec{\boldsymbol{z}}_{1:k-1}) = p(\vec{\boldsymbol{z}}_k|\vec{\boldsymbol{x}}_k)$

Objective

Probabilistically estimate $\vec{\boldsymbol{x}}_k$ using previous measurement $\vec{\boldsymbol{z}}_{1:k}$. In other words, construct the pdf $p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k})$.

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Optimal MMSE Estimate

$$\mathbb{E}\{\|\vec{\boldsymbol{x}}_k - \hat{\vec{\boldsymbol{x}}}_k\|^2\} = \int \|\vec{\boldsymbol{x}}_k - \hat{\vec{\boldsymbol{x}}}_k\|^2 p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k}) d\vec{\boldsymbol{x}}_k \quad (3)$$

in other words find the conditional mean

$$\hat{\vec{\boldsymbol{x}}}_k = \mathbb{E}\{\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k}\} = \int \vec{\boldsymbol{x}}_k p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k}) d\vec{\boldsymbol{x}}_k \quad (4)$$

Prediction Stage

$$p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k-1}) = \int p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{x}}_{k-1})\ p(\vec{\boldsymbol{x}}_{k-1}|\vec{\boldsymbol{z}}_{1:k-1})\ d\vec{\boldsymbol{x}}_{k-1} \tag{5}$$

Prediction Stage

$$p(\vec{\boldsymbol{x}}_k | \vec{\boldsymbol{z}}_{1:k-1}) = \int p(\vec{\boldsymbol{x}}_k | \vec{\boldsymbol{x}}_{k-1}) \; p(\vec{\boldsymbol{x}}_{k-1} | \vec{\boldsymbol{z}}_{1:k-1}) \; d\vec{\boldsymbol{x}}_{k-1} \tag{5}$$

defined using the state equation

Prediction Stage

$$p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k-1}) = \int p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{x}}_{k-1}) \; p(\vec{\boldsymbol{x}}_{k-1}|\vec{\boldsymbol{z}}_{1:k-1}) \; d\vec{\boldsymbol{x}}_{k-1} \tag{5}$$

Assumed available from previous iteration

Update Stage

Using Bayes' Rule

$$p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k}) = \frac{p(\vec{\boldsymbol{z}}_k|\vec{\boldsymbol{x}}_k)\;\; p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k-1})}{p(\vec{\boldsymbol{z}}_k|\vec{\boldsymbol{z}}_{1:k-1})} \tag{6}$$

Update Stage

Using Bayes' Rule

$$p(\vec{\mathbf{x}}_k|\vec{\mathbf{z}}_{1:k}) = \frac{p(\vec{\mathbf{z}}_k|\vec{\mathbf{x}}_k)\;\; p(\vec{\mathbf{x}}_k|\vec{\mathbf{z}}_{1:k-1})}{p(\vec{\mathbf{z}}_k|\vec{\mathbf{z}}_{1:k-1})} \qquad (6)$$

posterior

Update Stage

Using Bayes' Rule

$$p(\vec{\boldsymbol{x}}_k | \vec{\boldsymbol{z}}_{1:k}) = \frac{p(\vec{\boldsymbol{z}}_k | \vec{\boldsymbol{x}}_k)\, p(\vec{\boldsymbol{x}}_k | \vec{\boldsymbol{z}}_{1:k-1})}{p(\vec{\boldsymbol{z}}_k | \vec{\boldsymbol{z}}_{1:k-1})} \tag{6}$$

likelihood: defined by the measurement equation and the statistics of the measurement noise $\vec{\boldsymbol{v}}_k$

Update Stage

Using Bayes' Rule

$$p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k}) = \frac{p(\vec{\boldsymbol{z}}_k|\vec{\boldsymbol{x}}_k)\, p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k-1})}{p(\vec{\boldsymbol{z}}_k|\vec{\boldsymbol{z}}_{1:k-1})} \tag{6}$$

prior: defined by the state equation
and the statistics of the state noise $\vec{\boldsymbol{w}}_{k-1}$

Update Stage

Using Bayes' Rule

$$p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k}) = \frac{p(\vec{\boldsymbol{z}}_k|\vec{\boldsymbol{x}}_k)\ p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k-1})}{p(\vec{\boldsymbol{z}}_k|\vec{\boldsymbol{z}}_{1:k-1})} \tag{6}$$

evidence = $\int p(\vec{\boldsymbol{z}}_k|\vec{\boldsymbol{x}}_k)p(\vec{\boldsymbol{x}}_k|\vec{\boldsymbol{z}}_{1:k-1})d\vec{\boldsymbol{x}}_k$
and depends on the likelihood function $p(\vec{\boldsymbol{z}}_k|\vec{\boldsymbol{x}}_k)$

Limitations

1. Need to keep track of all previous states.
2. Generally can't be determined analytically.

Assumptions

- $\vec{w}_k$ and $\vec{v}_k$ are drawn from a Gaussian distribution, uncorrelated have zero mean and statistically independent.

$$\mathbb{E}\{\vec{w}_k \vec{w}_i^T\} = \begin{cases} \mathbf{Q}_k & i = k \\ 0 & i \neq k \end{cases} \tag{7}$$

$$\mathbb{E}\{\vec{v}_k \vec{v}_i^T\} = \begin{cases} \mathbf{R}_k & i = k \\ 0 & i \neq k \end{cases} \tag{8}$$

$$\mathbb{E}\{\vec{w}_k \vec{v}_i^T\} = \begin{cases} 0 & \forall i, k \end{cases} \tag{9}$$

Assumptions

- $\mathbf{f}_k$ and $\mathbf{h}_k$ are both linear, e.g., the state-space system equations may be written as

$$\vec{\boldsymbol{x}}_k = \boldsymbol{\Phi}_{k-1}\,\vec{\boldsymbol{x}}_{k-1} + \vec{\boldsymbol{w}}_{k-1} \tag{10}$$

$$\vec{\boldsymbol{y}}_k = \mathbf{H}_k\,\vec{\boldsymbol{x}}_k + \vec{\boldsymbol{v}}_k \tag{11}$$

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$(n \times n)$ transition matrix relating $\vec{\boldsymbol{x}}_{k-1}$ to $\vec{\boldsymbol{x}}_k$

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$(m \times n)$ matrix provides noiseless connection between measurement and state vectors

pdf notation

$$p(\vec{x}_{k-1}|\vec{z}_{1:k-1}) = \mathcal{N}(\vec{x}_{k-1}; \vec{m}_{k-1|k-1}, \ \mathbf{P}_{k-1|k-1}\ ) \tag{12}$$

$$p(\vec{x}_k|\vec{z}_{1:k-1}) = \mathcal{N}(\vec{x}_k; \vec{m}_{k|k-1}, \mathbf{P}_{k|k-1}) \tag{13}$$

$$p(\vec{x}_k|\vec{z}_{1:k}) = \mathcal{N}(\vec{x}_k; \vec{m}_{k|k}, \ \mathbf{P}_{k|k}\ ) \tag{14}$$

where

$$\vec{m}_{k|k-1} = \mathbf{\Phi}_{k-1}\vec{m}_{k-1|k-1} \tag{15}$$

$$\mathbf{P}_{k|k-1} = \mathbf{Q}_{k-1} + \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{\Phi}_{k-1}^T \tag{16}$$

$$\vec{m}_{k|k} = \vec{m}_{k|k-1} + \mathbf{K}_k(\vec{z}_k - \mathbf{H}_k\vec{m}_{k|k-1}) \tag{17}$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k|k-1} \tag{18}$$

pdf notation

a priori error covariance ($\mathbb{E}\{(\hat{\vec{x}}_{k-1|k-1} - \vec{x}_{k-1})(\hat{\vec{x}}_{k-1|k-1} - \vec{x}_{k-1})^T\}$).
Diagonal terms are the variances in the state estimates
off-diagonal show correlation between the errors in the different states

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$$p(\vec{\boldsymbol{x}}_{k}|\vec{\boldsymbol{z}}_{1:k}) = \mathcal{N}(\vec{\boldsymbol{x}}_{k}; \vec{\boldsymbol{m}}_{k|k}, \mathbf{P}_{k|k}) \tag{14}$$

posterior error covariance ($\mathbb{E}\{(\hat{\vec{\boldsymbol{x}}}_{k|k} - \vec{\boldsymbol{x}}_k)(\hat{\vec{\boldsymbol{x}}}_{k|k} - \vec{\boldsymbol{x}}_k)^T\}$)

where

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State-Space Equations

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$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1}\left(\mathbf{I} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k|k-1}\right) \tag{22}$$

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$(n \times m)$ Kalman gain

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Measurement innovation

Kalman Gain

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1} \tag{23}$$

Kalman Gain

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1} \quad (23)$$

Covariance of the innovation term

Kalman filter data flow

Initial estimate ($\hat{\vec{x}}_0$ and $\mathbf{P}_0$)

Kalman filter data flow

Initial estimate ($\hat{\mathbf{x}}_0$ and $\mathbf{P}_0$)

↓

Compute Kalman gain

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

Kalman filter data flow

Initial estimate ($\hat{\vec{x}}_0$ and $\mathbf{P}_0$)

↓

Compute Kalman gain
$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1}$$

↓

Update estimate with measurement $\vec{z}_k$
$$\hat{\vec{x}}_{k|k} = \hat{\vec{x}}_{k|k-1} + \mathbf{K}_k\left(\vec{z}_k - \mathbf{H}_k\hat{\vec{x}}_{k|k-1}\right)$$

Kalman filter data flow

Initial estimate ($\hat{\vec{\mathbf{x}}}_0$ and $\mathbf{P}_0$)

↓

Compute Kalman gain

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T\left(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1}$$

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Update estimate with measurement $\vec{\mathbf{z}}_k$

$$\hat{\vec{\mathbf{x}}}_{k|k} = \hat{\vec{\mathbf{x}}}_{k|k-1} + \mathbf{K}_k\left(\vec{\mathbf{z}}_k - \mathbf{H}_k\hat{\vec{\mathbf{x}}}_{k|k-1}\right)$$

↓

Update error covariance

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k|k-1}$$

Kalman filter data flow

Initial estimate ($\hat{\vec{x}}_0$ and $\mathbf{P}_0$)

Compute Kalman gain

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

Update estimate with measurement $\vec{z}_k$

$$\hat{\vec{x}}_{k|k} = \hat{\vec{x}}_{k|k-1} + \mathbf{K}_k(\vec{z}_k - \mathbf{H}_k\hat{\vec{x}}_{k|k-1})$$

Update error covariance

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k|k-1}$$

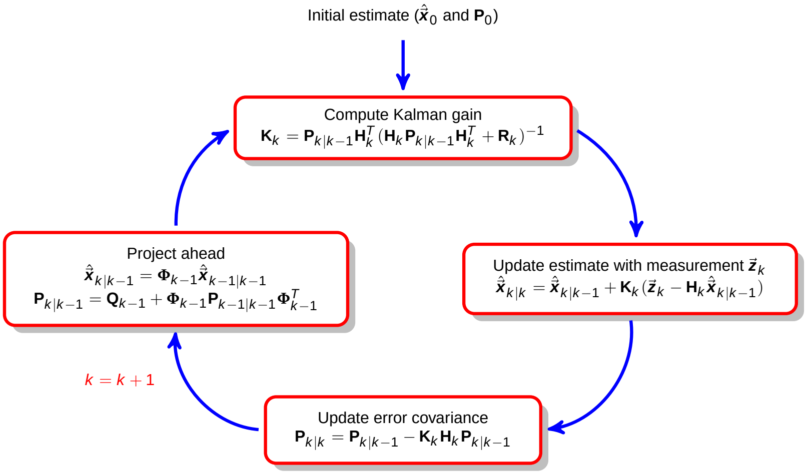
$k = k+1$

Project ahead

$$\hat{\vec{x}}_{k|k-1} = \mathbf{\Phi}_{k-1}\hat{\vec{x}}_{k-1|k-1}$$

$$\mathbf{P}_{k|k-1} = \mathbf{Q}_{k-1} + \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{\Phi}_{k-1}^T$$

Kalman filter data flow



Observability

The system is observable if the observability matrix

$$\mathcal{O}(k) = \begin{bmatrix} \boldsymbol{H}(k-n+1) \\ \boldsymbol{H}(k-n-2)\boldsymbol{\Phi}(k-n+1) \\ \vdots \\ \boldsymbol{H}(k)\boldsymbol{\Phi}(k-1)\ldots\boldsymbol{\Phi}(k-n+1) \end{bmatrix} \tag{24}$$

where n is the number of states, has a rank of n. The rank of $\mathcal{O}$ is a binary indicator and does **not** provide a measure of how close the system is to being unobservable, hence, is prone to numerical ill-conditioning.

A Better Observability Measure

In addition to the computation of the rank of $\mathcal{O}(k)$, compute the Singular Value Decomposition (SVD) of $\mathcal{O}(k)$ as

$$\mathcal{O} = U\Sigma V^* \tag{25}$$

and observe the diagonal values of the matrix Σ. Using this approach it is possible to monitor the variations in the system observability due to changes in system dynamics.

Remarks

- Kalman filter is optimal under the aforementioned assumptions,
- and it is also an unbiased and minimum variance estimate.
- If the Gaussian assumptions is not true, Kalman filter is biased and not minimum variance.
- Observability is dynamics dependent.
- The error covariance update may be implemented using the *Joseph form* which provides a more stable solution due to the guaranteed symmetry.

$$\boldsymbol{P}_{k|k} = (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{H}_k) \boldsymbol{P}_{k|k-1} (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{H}_k)^T + \boldsymbol{K}_k \boldsymbol{R}_k \boldsymbol{K}_k^T \quad (26)$$

System Model

$$\dot{\vec{x}}(t) = \mathbf{F}(t)\vec{x}(t) + \mathbf{G}(t)\vec{w}(t) \tag{27}$$

To obtain the state vector estimate $\hat{\vec{x}}(t)$

$$\mathbb{E}\{\dot{\vec{x}}(t)\} = \frac{\partial}{\partial t}\hat{\vec{x}}(t) = \mathbf{F}(t)\hat{\vec{x}}(t) \tag{28}$$

Solving the above equation over the interval $t - \tau_s, t$

$$\hat{\vec{x}}(t) = e^{\left(\int_{t-\tau_s}^{t} \mathbf{F}(t')dt'\right)}\hat{\vec{x}}(t - \tau_s) \tag{29}$$

where $\mathbf{F}_{k-1}$ is the average of $\mathbf{F}$ at times t and $t - \tau_s$.

System Model Discretization

As shown in the Kalman filter equations the state vector estimate is given by

$$\hat{\vec{\boldsymbol{x}}}_{k|k-1} = \boldsymbol{\Phi}_{k-1}\hat{\vec{\boldsymbol{x}}}_{k-1|k-1}$$

Therefore,

System Model Discretization

As shown in the Kalman filter equations the state vector estimate is given by

$$\hat{\vec{\mathbf{x}}}_{k|k-1} = \mathbf{\Phi}_{k-1}\hat{\vec{\mathbf{x}}}_{k-1|k-1}$$

Therefore,

$$\mathbf{\Phi}_{k-1} = e^{\mathbf{F}_{k-1}\tau_s} \approx \mathbf{I} + \mathbf{F}_{k-1}\tau_s \tag{30}$$

where $\mathbf{F}_{k-1}$ is the average of $\mathbf{F}$ at times t and $t - \tau_s$, and first order approximation is used.

Discrete Covariance Matrix $\mathbf{Q}_k$

The solution to (27) is

$$\vec{\boldsymbol{x}}_k = \boldsymbol{\Phi}_{k-1}\vec{\boldsymbol{x}}_{k-1} + \int_{t-\tau_s}^{t} e^{\mathbf{F}_{k-1}(t-\eta)}\mathbf{G}_{k-1}\vec{\boldsymbol{w}}(\eta)d\eta \tag{31}$$

where $\mathbf{G}_{k-1}$ is the average of $\mathbf{G}$ at times t and $t-\tau_s$. Now let's look at the error covariance matrix

$$\mathbf{P}_{k|k-1} = \mathbb{E}\{(\hat{\vec{\boldsymbol{x}}}_{k|k-1} - \vec{\boldsymbol{x}}_k)(\hat{\vec{\boldsymbol{x}}}_{k|k-1} - \vec{\boldsymbol{x}}_k)^T\} \tag{32}$$

where

$$\hat{\vec{\boldsymbol{x}}}_{k|k-1} - \vec{\boldsymbol{x}}_k = \boldsymbol{\Phi}_{k-1}(\hat{\vec{\boldsymbol{x}}}_{k-1|k-1} - \vec{\boldsymbol{x}}_k) - \int_{t-\tau_s}^{t} e^{\mathbf{F}_{k-1}(t-\eta)}\mathbf{G}_{k-1}\vec{\boldsymbol{w}}(\eta)d\eta \tag{33}$$

Discrete Covariance Matrix Q_k (cont.)

Since the state estimate errors and the system noise $\vec{w}(t)$ are uncorrelated

$$\begin{aligned}
\mathbf{P}_{k|k-1} =& \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{\Phi}_{k-1}^T + \\
& \mathbb{E}\left\{ \iint_{t-\tau_s}^{t} e^{F_{k-1}(t-\eta)}\mathbf{G}_{k-1}\vec{w}(\eta)\vec{w}^T(\zeta)\mathbf{G}_{k-1}^T e^{\mathbf{F}_{k-1}^T(t-\zeta)} d\eta d\zeta \right\} \\
=& \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{\Phi}_{k-1}^T + \\
& \iint_{t-\tau_s}^{t} e^{F_{k-1}(t-\eta)}\mathbf{G}_{k-1}\mathbb{E}\{\vec{w}(\eta)\vec{w}^T(\zeta)\}\mathbf{G}_{k-1}^T e^{\mathbf{F}_{k-1}^T(t-\zeta)} d\eta d\zeta \\
=& \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{\Phi}_{k-1}^T + \mathbf{Q}_{k-1}
\end{aligned} \tag{34}$$

Discrete Covariance Matrix $\mathbf{Q}_k$ (cont.)

Assuming white noise, small time step, **G** is constant over the integration period, and the trapezoidal integration

$$\mathbf{Q}_{k-1} \approx \frac{1}{2}\left[\mathbf{\Phi}_{k-1}\mathbf{G}_{k-1}\mathbf{Q}(t_{k-1})\mathbf{G}_{k-1}^T\mathbf{\Phi}_{k-1}^T + \mathbf{G}_{k-1}\mathbf{Q}(t_{k-1})\mathbf{G}_{k-1}^T\right]\tau_s \tag{35}$$

where

$$\mathbb{E}\{\vec{\mathbf{w}}(\eta)\vec{\mathbf{w}}^T(\zeta)\} = \mathbf{Q}(\eta)\delta(\eta - \zeta) \tag{36}$$

Linearized System

$$\mathbf{F}_k = \left.\frac{\partial \mathbf{f}(\vec{\boldsymbol{x}})}{\partial \vec{\boldsymbol{x}}}\right|_{\vec{\boldsymbol{x}}=\hat{\vec{\boldsymbol{x}}}_{k|k-1}}, \qquad \mathbf{H}_k = \left.\frac{\partial \mathbf{h}(\vec{\boldsymbol{x}})}{\partial \vec{\boldsymbol{x}}}\right|_{\vec{\boldsymbol{x}}=\hat{\vec{\boldsymbol{x}}}_{k|k-1}} \tag{37}$$

where

$$\frac{\partial \mathbf{f}(\vec{\boldsymbol{x}})}{\partial \vec{\boldsymbol{x}}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \\ \vdots & \ddots & \vdots \end{pmatrix}, \qquad \frac{\partial \mathbf{h}(\vec{\boldsymbol{x}})}{\partial \vec{\boldsymbol{x}}} = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots \\ \vdots & \ddots & \vdots \end{pmatrix} \tag{38}$$

First Order Markov Noise

State Equation

$$\dot{n}(t) = -\frac{1}{T_c} n(t) + w(t) \tag{39}$$

Autocorrelation Function

$$\mathbb{E}\{n(t)(n(t+\tau)\} = \sigma^2 e^{-|\tau|/T_c} \tag{40}$$

where

$$\mathbb{E}\{w(t)w(t+\tau)\} = Q(t)\delta(t-\tau) \tag{41}$$

$$Q(t) = \frac{2\sigma^2}{T_c} \tag{42}$$

and T_c is the correlation time.

Discrete First Order Markov Noise

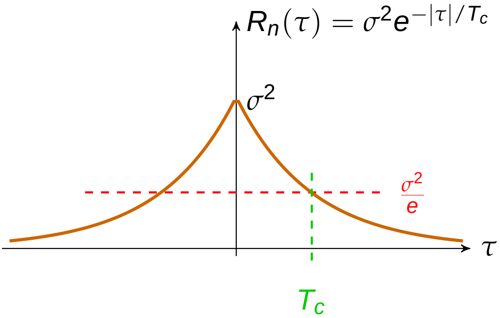
State Equation

$$n_k = e^{-\frac{1}{T_c}\tau_s} n_{k-1} + w_{k-1} \quad (43)$$

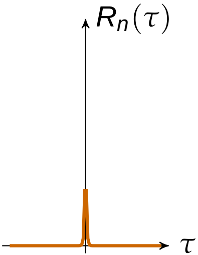
System Covariance Matrix

$$Q = \sigma^2 \left[1 - e^{-\frac{2}{T_c}\tau_s}\right] \quad (44)$$

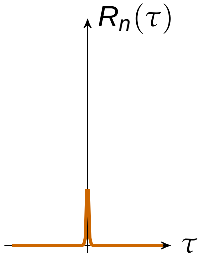
Autocorrelation of 1st order Markov

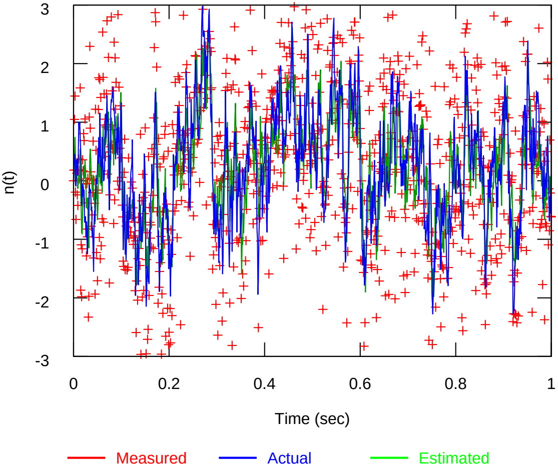


Small Correlation Time $T_c = 0.01$

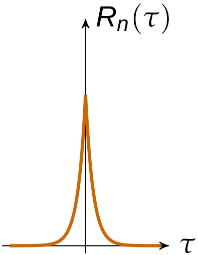


Small Correlation Time $T_c = 0.01$

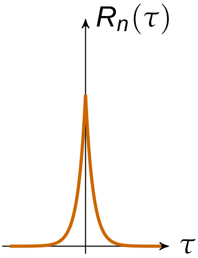


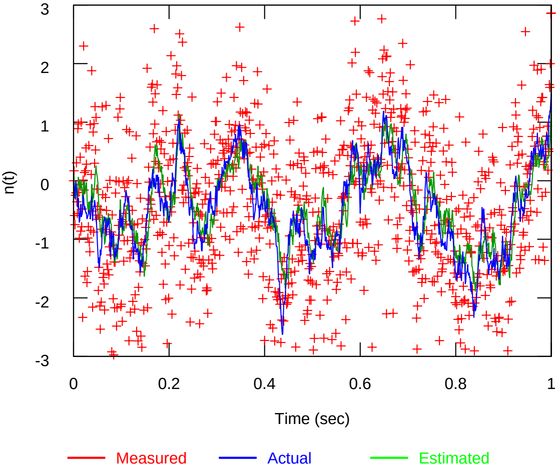


Larger Correlation Time $T_c = 0.1$



Larger Correlation Time $T_c = 0.1$





Unscented Kalman Filter (UKF)

Propagates carefully chosen sample points (using unscented transformation) through the true non-linear system, and therefore captures the posterior mean and covariance accurately to the second order.

Particle Filter

A Monte Carlo based method. It allows for a complete representation of the state distribution function. Unlike EKF and UKF, particle filters do not require the Gaussian assumptions.

Bayesian Filtering: From Kalman Filters to Particle Filters, and Beyond, by Zhe Chen