Curvelinear Analysis

- **Some notation definitions needed to use "intuitive" variable names**

Define a few Variables

R_0 is the earth’s semi-major axis ~6.378137×10^6 meters

R_E ir the transverse radius of curvature

$$R_E = \frac{R_0}{\sqrt{1 - e^2\, \mathrm{Sin}[L_b]^2}};$$

R_N ir the meridian radius of curvature

$$R_N = \frac{R_0\left(1 - e^2\right)}{\left(1 - e^2\, \mathrm{Sin}[L_b]^2\right)^{\frac{3}{2}}};$$

Previously derived (Eqn. 2.70), body position in the ECEF frame, described in the ECEF frame:

$$\vec{r}^{\,e}_{e,b} = \begin{pmatrix} (R_E + h_b)\, \mathrm{Cos}[L_b]\, \mathrm{Cos}[\lambda_b] \\ (R_E + h_b)\, \mathrm{Cos}[L_b]\, \mathrm{Sin}[\lambda_b] \\ \left(\left(1 - e^2\right) R_E + h_b\right) \mathrm{Sin}[L_b] \end{pmatrix};$$

Compute the Velocity : $\vec{v}^{\,e}_{e,b}$ =

$$\frac{d\,\vec{r}^{\,e}_{e,b}}{dt} = \frac{\delta\,\vec{r}^{\,e}_{e,b}}{\delta L_b}\frac{dL_b}{dt} + \frac{\delta\,\vec{r}^{\,e}_{e,b}}{\delta \gamma_b}\frac{d\gamma_b}{dt} + \frac{\delta\,\vec{r}^{\,e}_{e,b}}{\delta h_b}\frac{dh_b}{dt}$$

- **First compute $\frac{\delta\,\vec{r}^{\,e}_{e,b}}{\delta L_b}$**

$$\mathrm{Print}\left["\frac{\delta\,\vec{r}^{\,e}_{e,b}}{\delta L_b}=", \mathrm{MatrixForm}\left[\mathrm{Simplify}\left[D\left[\vec{r}^{\,e}_{e,b}, L_b\right]\right]\right] \;/.\; \left\{\mathrm{Cos}[2\,L_b] \to 1 - 2\,\mathrm{Sin}[L_b]^2\right\}\right]$$

$$\frac{\delta\,\vec{r}^{\,e}_{e,b}}{\delta L_b} = \begin{pmatrix} -\frac{\mathrm{Cos}[\lambda_b]\,\mathrm{Sin}[L_b]\left(\sqrt{2}\,\left(2-e^2+e^2\,\left(1-2\,\mathrm{Sin}[L_b]^2\right)\right)^{3/2} h_b+4\,R_0-4\,e^2\,R_0\right)}{4\,\left(1-e^2\,\mathrm{Sin}[L_b]^2\right)^{3/2}} \\ -\frac{\mathrm{Sin}[L_b]\,\mathrm{Sin}[\lambda_b]\left(\sqrt{2}\,\left(2-e^2+e^2\,\left(1-2\,\mathrm{Sin}[L_b]^2\right)\right)^{3/2} h_b+4\,R_0-4\,e^2\,R_0\right)}{4\,\left(1-e^2\,\mathrm{Sin}[L_b]^2\right)^{3/2}} \\ -\frac{\mathrm{Cos}[L_b]\left(-\frac{\left(2-e^2+e^2\,\left(1-2\,\mathrm{Sin}[L_b]^2\right)\right)^{3/2} h_b}{2\,\sqrt{2}}+\left(-1+e^2\right) R_0\right)}{\left(1-e^2\,\mathrm{Sin}[L_b]^2\right)^{3/2}} \end{pmatrix}$$

Looking at the top and midle terms

```
Simplify[ (√2 (2 - e^2 + e^2 (1 - 2 Sin[L_b]^2))^(3/2) h_b) / (4 (1 - e^2 Sin[L_b]^2)^(3/2)) ]
```

h_b

$$\therefore \text{Term}_1 = -\text{Cos}[\lambda_b]\,\text{Sin}[L_b]\left(h_b + \frac{R_0\,(1-e^2)}{(1-e^2\,\text{Sin}[L_b]^2)^{\frac{3}{2}}}\right) = -\text{Cos}[\lambda_b]\,\text{Sin}[L_b]\,(h_b + R_N)$$

$$\therefore \text{Term}_2 = -\text{Sin}[L_b]\,\text{Sin}[\lambda_b]\left(h_b + \frac{R_0\,(1-e^2)}{(1-e^2\,\text{Sin}[L_b]^2)^{\frac{3}{2}}}\right) = -\text{Sin}[L_b]\,\text{Sin}[\lambda_b]\,(h_b + R_N)$$

Looking at the bottom term

```
Simplify[ ((2 - e^2 + e^2 (1 - 2 Sin[L_b]^2))^(3/2) h_b) / (2 √2) ]
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$(1 - e^2\,\text{Sin}[L_b]^2)^{3/2}\,h_b$

$$\therefore \text{Term}_3 = -\text{Cos}[L_b]\left(\frac{-(1-e^2\,\text{Sin}[L_b]^2)^{3/2}\,h_b - (1-e^2)\,R_0}{(1-e^2\,\text{Sin}[L_b]^2)^{\frac{3}{2}}}\right) = \text{Cos}[L_b]\,(h_b + R_N)$$

$$\frac{\delta\,\vec{r}^{\,e}_{e,b}}{\delta L_b} = (R_N + h_b)\begin{pmatrix} -\text{Cos}[\lambda_b]\,\text{Sin}[L_b] \\ -\text{Sin}[L_b]\,\text{Sin}[\lambda_b] \\ \text{Cos}[L_b] \end{pmatrix}$$

■ Next compute $\frac{\delta\,\vec{r}^{\,e}_{e,b}}{\delta\gamma_b}$

```
Print["δr⃗^e_e,b/δλ_b =", MatrixForm[Simplify[D[r⃗^e_e,b, λ_b]]] /. {R_0/√(1 - e^2 Sin[L_b]^2) → "R_E"}]
```

$$\frac{\delta\,\vec{r}^{\,e}_{e,b}}{\delta\lambda_b} = \begin{pmatrix} -\text{Cos}[L_b]\,\text{Sin}[\lambda_b]\,(R_E + h_b) \\ \text{Cos}[L_b]\,\text{Cos}[\lambda_b]\,(R_E + h_b) \\ 0 \end{pmatrix}$$

- **Next compute** $\frac{\delta \vec{r}^{e}_{e,b}}{\delta \gamma_b}$

```
Print["δr⃗ᵉ_{e,b}/δλ_b =", MatrixForm[Simplify[D[r⃗ᵉ_{e,b}, λ_b]]] /. {R0/Sqrt[1 - e^2 Sin[L_b]^2] → "R_E"}]
```

$$\frac{\delta \vec{r}^{e}_{e,b}}{\delta \lambda_b} = \begin{pmatrix} -\text{Cos}[L_b]\,\text{Sin}[\lambda_b]\,(R_E + h_b) \\ \text{Cos}[L_b]\,\text{Cos}[\lambda_b]\,(R_E + h_b) \\ 0 \end{pmatrix}$$

$$\frac{\delta \vec{r}^{e}_{e,b}}{\delta \lambda_b} = (R_E + h_b)\,\text{Cos}[L_b] \begin{pmatrix} -\text{Sin}[\lambda_b] \\ \text{Cos}[\lambda_b] \\ 0 \end{pmatrix}$$

- **Finally compute** $\frac{\delta \vec{r}^{e}_{e,b}}{\delta h_b}$

```
Print["δr⃗ᵉ_{e,b}/δh_b =", MatrixForm[Simplify[D[r⃗ᵉ_{e,b}, h_b]]] ]
```

$$\frac{\delta \vec{r}^{e}_{e,b}}{\delta h_b} = \begin{pmatrix} \text{Cos}[L_b]\,\text{Cos}[\lambda_b] \\ \text{Cos}[L_b]\,\text{Sin}[\lambda_b] \\ \text{Sin}[L_b] \end{pmatrix}$$

$$\frac{\delta \vec{r}^{e}_{e,b}}{\delta h_b} = \begin{pmatrix} \text{Cos}[L_b]\,\text{Cos}[\lambda_b] \\ \text{Cos}[L_b]\,\text{Sin}[\lambda_b] \\ \text{Sin}[L_b] \end{pmatrix}$$

- **Composing the final result :** $\vec{v}^{e}_{e,b} =$

$$\frac{d\,\vec{r}^{e}_{e,b}}{dt} = \frac{\delta \vec{r}^{e}_{e,b}}{\delta L_b}\frac{dL_b}{dt} + \frac{\delta \vec{r}^{e}_{e,b}}{\delta \gamma_b}\frac{d\gamma_b}{dt} + \frac{\delta \vec{r}^{e}_{e,b}}{\delta h_b}\frac{dh_b}{dt} = \begin{pmatrix} \frac{\delta \vec{r}^{e}_{e,b}}{\delta L_b} & \frac{\delta \vec{r}^{e}_{e,b}}{\delta \gamma_b} & \frac{\delta \vec{r}^{e}_{e,b}}{\delta h_b} \end{pmatrix} \begin{pmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{pmatrix}$$

$$\vec{v}^{e}_{e,b} = \begin{pmatrix} -\text{Cos}[\lambda_b]\,\text{Sin}[L_b] & -\text{Sin}[\lambda_b] & -\text{Cos}[L_b]\,\text{Cos}[\lambda_b] \\ -\text{Sin}[L_b]\,\text{Sin}[\lambda_b] & \text{Cos}[\lambda_b] & -\text{Cos}[L_b]\,\text{Sin}[\lambda_b] \\ \text{Cos}[L_b] & 0 & -\text{Sin}[L_b] \end{pmatrix}$$

$$\begin{pmatrix} (R_N + h_b)\,\dot{L}_b \\ \text{Cos}[L_b]\,(R_E + h_b)\,\dot{\lambda}_b \\ -\dot{h}_b \end{pmatrix}$$

Comparing the matrix with Eqn 2.99 (i.e. C^{e}_{n})

$$\vec{v}^{e}_{e,b} = C^{e}_{n} \begin{pmatrix} (R_N + h_b)\,\dot{L}_b \\ \mathrm{Cos}[L_b]\,(R_E + h_b)\,\dot{\lambda}_b \\ -\dot{h}_b \end{pmatrix} = C^{e}_{n}\,\vec{v}^{n}_{e,b}$$

Which suggests that

$$\vec{v}^{n}_{e,b} = \begin{pmatrix} (R_N + h_b)\,\dot{L}_b \\ \mathrm{Cos}[L_b]\,(R_E + h_b)\,\dot{\lambda}_b \\ -\dot{h}_b \end{pmatrix}$$