Lecture

Power Spectral Density Estimation

EE 570: Location and Navigation

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The purpose is to estimate the distribution of power in a signal. Unfortunately, truth and what is practical cause a problem.

Truth

- Infinitely long.
- Continuous in time and value.
- Provides true distribution of power.

Practice

- Finite length.
- Discrete in time and value.
- Only approximation of distribution of power.

Let's make it more interesting
The signal is stochastic in nature.

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1 Review Material

1.1 Signal Classification

Assume the voltage across a resistor R is $e(t)$ and is producing a current $i(t)$. The instantaneous power per ohm is $p(t) = e(t)i(t)/R = i^2(t)$.

Total Energy

$$E = \lim_{T\to\infty} \int_{-T}^{T} i^2(t)dt \tag{1}$$

Average Power

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)dt \tag{2}$$

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Arbitrary signal $x(t)$

Total Normalized Energy

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{3}$$

Normalized Power

$$P \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{4}$$

- $x(t)$ is an *energy signal* iff $0 < E < \infty$, so that $P = 0$.
- $x(t)$ is a *power signal* iff $0 < P < \infty$, so that $E = \infty$.

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1.2 Time Averages

For Energy Signals

$$\phi(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \tag{5}$$

Provides a measure of similarity or coherence between a signal and a delayed version of itself. Note that $\phi(0) = E$

For Power Signals

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt \tag{6}$$

For Periodic Signals

$$R(\tau) = \frac{1}{T_0} \int_{T_0} x(t)x(t+\tau)dt \tag{7}$$

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1.3 Frequency Domain

Energy Spectral Density

Rayleigh's Energy Theorem or Parseval's theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF \tag{8}$$

Energy Spectral Density

$$G(F) \triangleq |X(F)|^2 \tag{9}$$

with units of *volts*^{2}-*sec*^{2} or, if considered on a per-ohm basis, *watts-sec/Hz=joules/Hz*

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Power Spectral Density

$$P = \int_{-\infty}^{\infty} S(F)dF = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{10}$$

where we define $S(F)$ as the power spectral density with units of watts/Hz. Note that $R(0) = \int_{-\infty}^{\infty} S(F)dF$. .7

2 Random Signals and Noise

Basic Definitions

- Define an *experiment* with random *outcome*.
- Mapping of the outcome to a variable ⇒ random variable.
- Mapping of the outcome to a function ⇒ random function.

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2.1 Statistical Averages

Probability (Cumulative) Distribution Function (cdf)

$$F_X(x) = \texttt{probability that } X \leq x = P(X \leq x) \tag{11}$$

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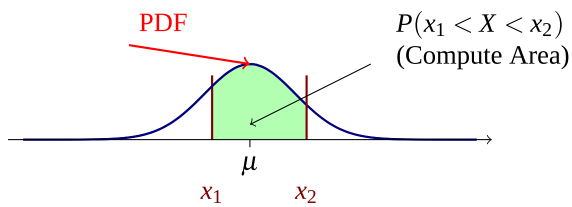
Probability Density Function (pdf)

$$f_X(x) = \frac{dF_X(x)}{dx} \tag{12}$$

and

$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx \tag{13}$$



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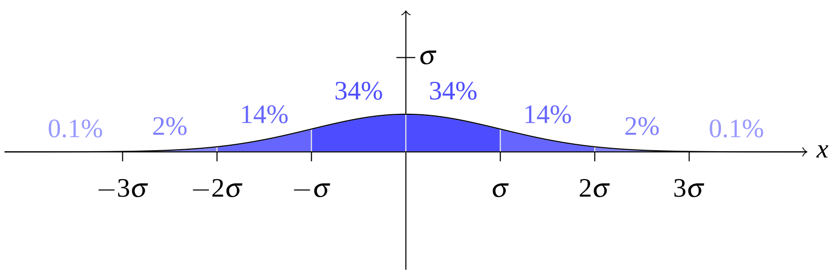
Gaussian Distribution

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$



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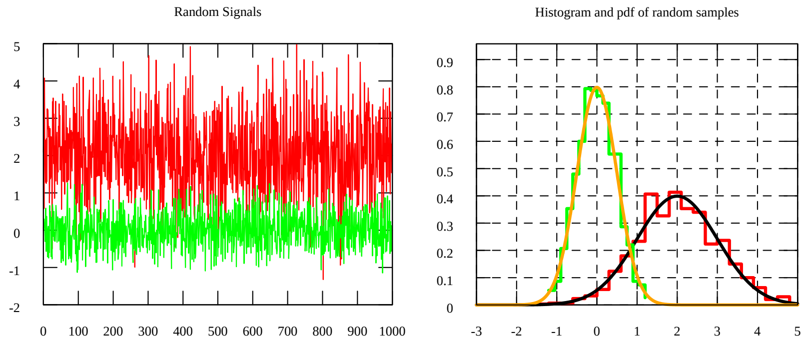
PDF of White Noise



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Mean of a Discrete RV

$$\bar{X} = \mathbb{E}[X] = \sum_{j=1}^{M} x_j P_j \tag{14}$$

Mean of a Continuous RV

$$\bar{X} = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{15}$$

Variance of a RV

$$\sigma_X^2 \triangleq \mathbb{E}\left\{[X - \mathbb{E}(X)]^2\right\} = \mathbb{E}[X^2] - \mathbb{E}^2[X] \tag{16}$$

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Given a two random variables X and Y.

Covariance

$$\mu_{XY} = \mathbb{E}\left\{[X - \bar{x}][Y - \bar{Y}]\right\} = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \tag{17}$$

Correlation Coefficient

$$\rho_{XY} = \frac{\mu_{XY}}{\sigma_X \sigma_Y} \tag{18}$$

Autocorrelation

$$\Gamma_X(\tau) = \mathbb{E}[X(t)X(t+\tau)] \tag{19}$$

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2.2 Stochastic Processes

Terminology

See Figure 1

- $X(t, \zeta_i)$: sample function.
- The governing experiment: random or stochastic process.
- All sample functions: ensemble.
- $X(t_j, \zeta)$: random variable.

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Strict Sense Stationarity

If the joint pdfs depend only on the time difference regardless of the time origin, then the random process is known as *stationary*.

For stationary process means and variances are independent of time and the covariance depends only on the time difference. .16

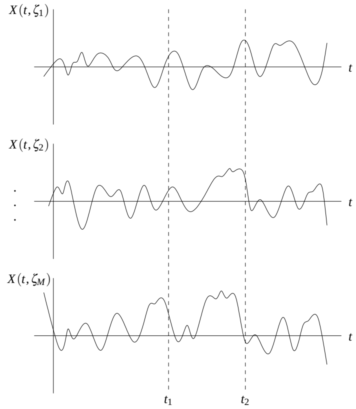

Figure 1: Sample functions of a random process

Wide Sense Stationarity

If the joint pdfs depends on the time difference but the mean and variances are time-independent, then the random process is known as *wide-sense-stationary*.

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Ergodicity

If the time statistics equals ensemble statistics, then the random process is known as *ergodic*.

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2.3 Correlation and Power Spectral Density

Power Spectral Density

Given a sample function $X(t,\zeta_i)$ of a random process, we first obtain the power spectral density by means of the Fourier transform of a truncated version $X_T(t,\zeta_i)$ defined as

$$X_T(t,\zeta_i) = \begin{cases} X(t,\zeta_i), & |t| < \frac{1}{2}T \\ 0, & \texttt{otherwise} \end{cases} \tag{20}$$

The Fourier transform of $X_T(t,\zeta_i)$ is

$$\mathscr{F}\{X_T(t,\zeta_i)\} = \int_{-T/2}^{T/2} X(t,\zeta_i)e^{j2\pi Ft}dt \tag{21}$$

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Power Spectral Density of a Random Process

The energy spectral density is $|\mathscr{F}\{X_T(t,\zeta_i)\}|^2$ and the average power density over the T is $|\mathscr{F}\{X_T(t,\zeta_i)\}|^2/T$. Since we have many sample functions, it is intuitive to take the ensemble average as $T \to \infty$, therefor the power spectral density, $S_X(F)$ is given by

$$S_X(F) = \lim_{T\to\infty} \frac{\overline{|\mathscr{F}\{X_T(t,\zeta_i)\}|^2}}{T} \tag{22}$$

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Wiener-Khinchine Theorem

$$S_X(F) = \lim_{T\to\infty} \int_{-2T}^{2T} \left(1 - \frac{|u|}{2T}\right) \Gamma_X(u) e^{-j\Omega u} du \tag{23}$$

as $T \to \infty$

$$S(F) \stackrel{\mathscr{F}}{\longleftrightarrow} \Gamma(\tau) \tag{24}$$

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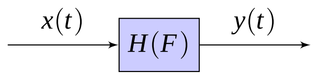
2.4 Input-Output Relationship of Linear Systems



$$S_Y(F) = |H(F)|^2 S_X(F) \tag{25}$$

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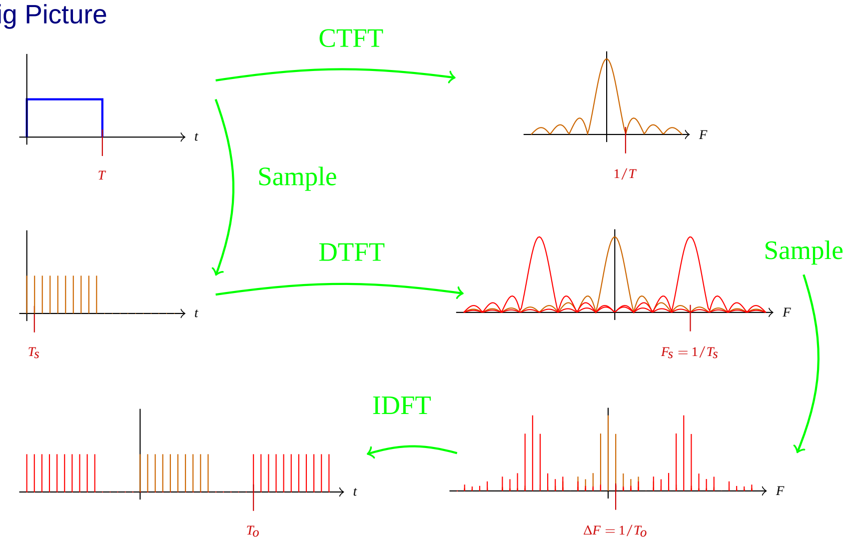
3 Discrete Signals and Systems

Big Picture



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Sampling Remarks

- Must sample more than twice bandwidth to avoid aliasing.
- FFT represents a periodic version of the time domain signal $\rightarrow$ could have time domain aliasing.
- Number of points in FFT is the same as number of points in time domain signal.

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4 Power Spectral Density

Obtaining PSD for Discrete Signals

What we want is

$$\Gamma_X(\tau) = \mathbb{E}[X(t)X(t+\tau)] \xrightarrow{\mathscr{CTFT}} S_X(F)$$

For infinitely long signals.

What we can compute is

$$\gamma_X(m) = \mathbb{E}[X(n)X(n+m)] \xrightarrow{\mathscr{DFT}} P_X(f)$$

For finite length signals.

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What do we need in an estimate

As $N \to \infty$ and in the mean squared sense

Unbiased
Asymptotically the mean of the estimate approaches the true power.

Variance
Variance of the estimate approaches zero.

Resulting in a consistent estimate of the power spectrum.

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Possible PSD Options

Periodogram
computed using $1/N$ times the magnitude squared of the FFT

$$\lim_{N\to\infty} \mathbb{E}[P_X(f)] = S_X(f)$$

$$\lim_{N\to\infty} var[P_X(f)] = S_X^2(f)$$

Welch Method
computed by segmenting the data (allowing overlaps), windowing the data in each segment then computing the average of the resultant priodogram

$$\mathbb{E}[P_X(f)] = \frac{1}{2\pi MU} S_X(f) \circledast W(f)$$

$$var[P_X(f)] \approx \frac{9}{8L} S_X^2(f)$$

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Welch Method

Assuming data length N, segment length M, Bartlett window, and 50% overlap

- FFT length = $M = 1.28/\Delta f = 1.28 F_s/\Delta F$
- Resulting number of segments = $L = \frac{2N}{M}$
- Length of data collected in sec. = $\frac{1.28L}{2\Delta F}$

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pwelch Function

```
[Pxx,f] = pwelch(x,window,noverlap,...
                 nfft,fs,'range')
```

You can use [] in fields that you want the default to be used.

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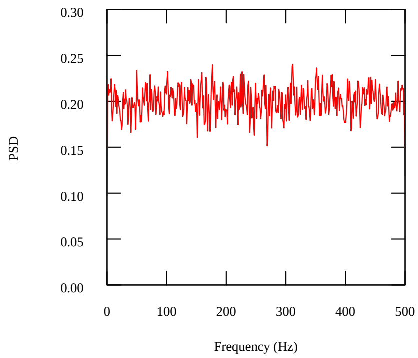
pwelch Function - WGN signal

```
Fs = 1000;
x = sqrt(0.1*Fs)*randn(1,100000);
[Pxx,f] = pwelch(x,1024,[],[],Fs,'onesided');
```



- Variance to high.

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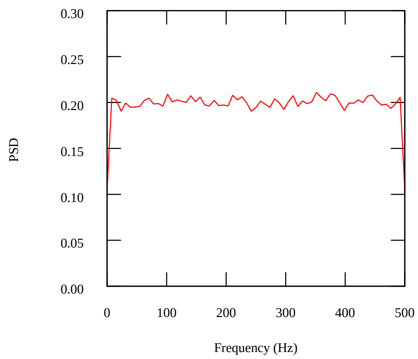
pwelch Function - WGN signal

```
[Pxx,f] = pwelch(x,128,[],[],Fs,'onesided')
```



- Reduced window size.
- Variance is now smaller.

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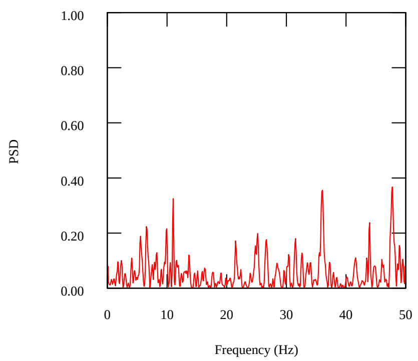
pwelch Function - cos + WGN signal

```
Fs = 100;   t = 0:1/Fs:5;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,1024,[],[],Fs,'onesided');
```



- Window larger than length of data.
- Frequency components can't be resolved.
- Variance high.

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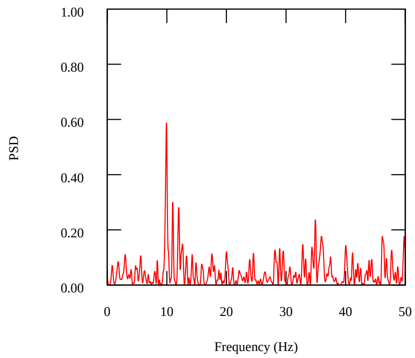
pwelch Function - cos + WGN signal

```
Fs = 100;   t = 0:1/Fs:5;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,1024,[],4096,Fs,'onesided');
```



- As expected increasing `nFFT` does not help.

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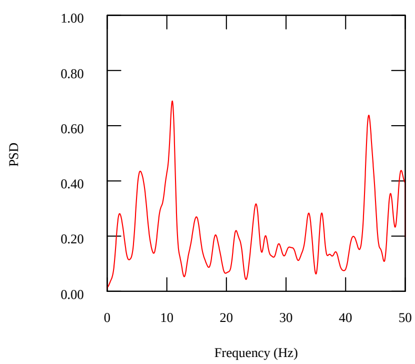
pwelch Function - cos + WGN signal

```
Fs = 100;   t = 0:1/Fs:5;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,128,[],4096,Fs,'onesided');
```



- Decreasing the window size decreases the variance.
- Still can't resolve the two frequencies.

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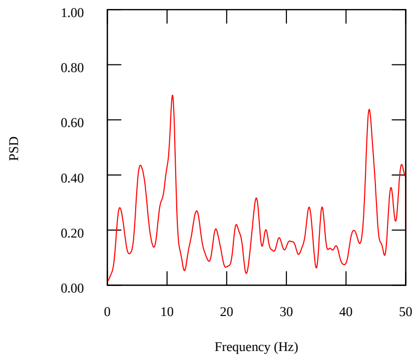
pwelch Function - cos + WGN signal

```
Fs = 100;   t = 0:1/Fs:50;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,128,[],4096,Fs,'onesided');
```



- Length of data sequence must be increased.
- Still can't resolve the two frequencies as the window size is too small.

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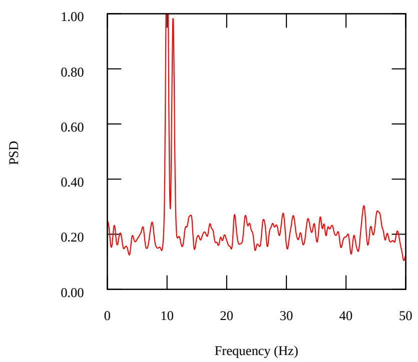
pwelch Function - cos + WGN signal

```
Fs = 100;   t = 0:1/Fs:50;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,256,[],4096,Fs,'onesided');
```



- Now we can resolve the two frequencies.

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Spectral Estimation - Remarks

- The length of the data sequence determines the maximum resolution that can be observed.
- Increasing the window length of each segment in the data increases the resolution.
- Decreasing the window length of each segment in the data decreases the variance of the estimate.
- `nFFT` only affects the amount of details shown and not the resolution.

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