EE 570: Location and Navigation

Error Mechanization (ECI)

Aly El-Osery   Kevin Wedeward

Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA

In Collaboration with
Stephen Bruder
Electrical and Computer Engineering Department
Embry-Riddle Aeronautical Univesity
Prescott, Arizona, USA

March 31, 2016

We have already derived the kinematic models in several frames. These models may be written in the form

$$\dot{\vec{x}} = \mathrm{f}(\vec{x}, \vec{u}) \tag{1}$$

where f is possibly non-linear.

Due to errors in the measurements we estimate $\vec{x}$ by integrating

$$\dot{\hat{\vec{x}}} = \mathrm{f}(\hat{\vec{x}}, \hat{\vec{u}}) \tag{2}$$

where $\hat{\vec{u}}$ is the measurement vector from the sensors after applying calibration corrections.

Due to errors in the measurements we estimate $\vec{x}$ by integrating

$$\dot{\hat{\vec{x}}} = \mathrm{f}(\hat{\vec{x}}, \hat{\vec{u}}) \tag{2}$$

where $\hat{\vec{u}}$ is the measurement vector from the sensors after applying calibration corrections.

If somehow we can model and possibly measure the error in the state we can then subtract it from the estimate to obtain an accurate position, velocity and attitude. We may also want to linearize the problem so that linear estimation approaches could be used.

All accelerometers and gyroscopes suffer from

- Biases
- Scale factor
- Cross-coupling
- Random noise

- *Fixed Errors*: deterministic and are present all the time, hence can be removed using calibration.
- *Temperature Dependent*: variations dependent on temperature and also may be modeled and characterized during calibration.
- *Run-to-run*: changes in the sensor error every time the sensor is run and is random in nature.
- *In-run*: random variations as the sensor is running.

Notation Used

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

Nothing above

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

Use "tilde"

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

"Use hat"

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$
Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{3}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$
Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{3}$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \ & f(\hat{\vec{x}}, t) \ + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ \approx \ & \dot{\hat{\vec{x}}} \ + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$
Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{3}$$

Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}}, t) + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T$$

$$\approx \dot{\hat{\vec{x}}} + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$
Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{3}$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = &\ f(\hat{\vec{x}}, t) \ + \left.\frac{\partial f(\vec{x}, t)}{\partial\vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ \approx &\ \dot{\hat{\vec{x}}} \ + \left.\frac{\partial f(\vec{x}, t)}{\partial\vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

$$\Rightarrow \delta\dot{\vec{x}} \approx \left.\frac{\partial f(\vec{x}, t)}{\partial\vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \tag{4}$$

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}^{\,b}_{ib}$ and $\tilde{\vec{\omega}}^{\,b}_{ib}$, respectively, will be modeled as

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta\vec{f}^{\,b}_{ib} \tag{5}$$

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{\omega}^{\,b}_{ib} + \Delta\vec{\omega}^{\,b}_{ib} \tag{6}$$

where $\vec{f}^{\,b}_{ib}$ and $\vec{\omega}^{\,b}_{ib}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}^{\,b}_{ib}$ and $\Delta\vec{\omega}^{\,b}_{ib}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}\,^b_{ib}$ and $\tilde{\vec{\omega}}\,^b_{ib}$, respectively, will be modeled as

$$\tilde{\vec{f}}\,^b_{ib} = \vec{f}\,^b_{ib} + \Delta\vec{f}\,^b_{ib} \tag{5}$$

$$\tilde{\vec{\omega}}\,^b_{ib} = \vec{\omega}\,^b_{ib} + \Delta\vec{\omega}\,^b_{ib} \tag{6}$$

these terms may be expanded further

where $\vec{f}\,^b_{ib}$ and $\vec{\omega}\,^b_{ib}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}\,^b_{ib}$ and $\Delta\vec{\omega}\,^b_{ib}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

Accelerometers

$$\tilde{\vec{f}}\,^b_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}\,^b_{ib} + \vec{nl}_a + \vec{w}_a \tag{7}$$

Gyroscopes

$$\tilde{\vec{\omega}}\,^b_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}\,^b_{ib} + G_g\vec{f}\,^b_{ib} + \vec{w}_g \tag{8}$$

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \quad (7)$$

Biases

Gyroscopes

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{\,b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \quad (8)$$

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \quad (7)$$

Misalignment and SF Errors

Gyroscopes

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{\,b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \quad (8)$$

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{7}$$

Non-linearity

Gyroscopes

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{\,b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{8}$$

NEW MEXICO TECH
SCIENCE • ENGINEERING • RESEARCH UNIVERSITY

Accelerometers

$$\tilde{\vec{f}}\,^b_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}\,^b_{ib} + \vec{nl}_a + \vec{w}_a \tag{7}$$

G-Sensitivity

Gyroscopes

$$\tilde{\vec{\omega}}\,^b_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}\,^b_{ib} + G_g\vec{f}\,^b_{ib} + \vec{w}_g \tag{8}$$

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{7}$$

Noise

Gyroscopes

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{\,b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{8}$$

Define the error state vector as

$$\delta\vec{x}^{\gamma}_{INS} = \begin{pmatrix} \delta\vec{\psi}^{\gamma}_{\gamma b} \\ \delta\vec{v}^{\gamma}_{\beta b} \\ \delta\vec{r}^{\gamma}_{\beta b} \end{pmatrix}, \quad \gamma, \beta \in i, e, n \tag{9}$$

Think of $\delta\vec{x}$ as the truth minus the estimate, i.e.,

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}} \tag{10}$$

The subtraction doesn't apply to the attitude component of the vector and needs to be treated differently

- Position error

$$\delta\vec{r}^{\gamma}_{\beta b} = \vec{r}^{\gamma}_{\beta b} - \hat{\vec{r}}^{\gamma}_{\beta b} \tag{11}$$

- Velocity error

$$\delta\vec{v}^{\gamma}_{\beta b} = \vec{v}^{\gamma}_{\beta b} - \hat{\vec{v}}^{\gamma}_{\beta b} \tag{12}$$

- Specific force errors

$$\delta\vec{f}^{b}_{ib} = \vec{f}^{b}_{ib} - \hat{\vec{f}}^{b}_{ib} \tag{13}$$

$$\Delta_e\vec{f}^{b}_{ib} = \Delta\vec{f}^{b}_{ib} - \Delta\hat{\vec{f}}^{b}_{ib} = -\delta\vec{f}^{b}_{ib} \tag{14}$$

- Angular rate errors

$$\delta\vec{\omega}^{b}_{ib} = \vec{\omega}^{b}_{ib} - \hat{\vec{\omega}}^{b}_{ib} \tag{15}$$

$$\Delta_e\vec{\omega}^{b}_{ib} = \Delta\vec{\omega}^{b}_{ib} - \Delta\hat{\vec{\omega}}^{b}_{ib} = -\delta\vec{\omega}^{b}_{ib} \tag{16}$$

Define

$$\delta C_b^\gamma = C_b^\gamma \hat{C}_\gamma^b = e^{[\delta\vec{\psi}^{\gamma}_{\gamma b}\times]} \approx \mathcal{I} + [\delta\vec{\psi}^{\gamma}_{\gamma b}\times] \tag{17}$$

This is the error in attitude resulting from errors in estimating the angular rates.

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])C_b^\gamma \tag{18}$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])\hat{C}_b^\gamma \tag{19}$$

Similarly the measuremed specific force and angular rate may be written in terms of the estimates as

$$\tilde{\vec{f}}\,{}^{b}_{ib} = \hat{\vec{f}}\,{}^{b}_{ib} + \Delta\hat{\vec{f}}\,{}^{b}_{ib} \quad (20)$$

$$\tilde{\vec{\omega}}\,{}^{b}_{ib} = \hat{\vec{\omega}}\,{}^{b}_{ib} + \Delta\hat{\vec{\omega}}\,{}^{b}_{ib} \quad (21)$$

where $\hat{\vec{f}}\,{}^{b}_{ib}$ and $\hat{\vec{\omega}}\,{}^{b}_{ib}$ are the accelerometer and gyroscope estimated calibration values, respectively.

Since the sensor measurements are corrupted with errors, derive an error model describing the position, velocity, and attitude as a function of time.

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b = \frac{d}{dt}\left[\left(\mathcal{I} + [\delta \vec{\psi}\,_{ib}^{i} \times] \right) \hat{C}_b^i \right] =$$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b \;= \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i\Omega_{ib}^b = [\delta\dot{\vec{\psi}}^{\,i}_{ib}\times]\hat{C}_b^i + (\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\dot{\hat{C}}_b^i =$$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i\Omega_{ib}^b = [\delta\dot{\vec{\psi}}^{\,i}_{ib}\times]\hat{C}_b^i + (\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\dot{\hat{C}}_b^i =$$

$$(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i(\hat{\Omega}_{ib}^b + \delta\Omega_{ib}^b) =$$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i\Omega_{ib}^b = [\delta\dot{\vec{\psi}}^{\,i}_{ib}\times]\hat{C}_b^i + (\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\dot{\hat{C}}_b^i =$$

$$(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i(\hat{\Omega}_{ib}^b + \delta\Omega_{ib}^b) =$$

$$= \hat{C}_b^i\delta\Omega_{ib}^b + (\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i\hat{\Omega}_{ib}^b$$

$$\because [\delta\vec{\psi}^{\,i}_{ib}\times]\delta\Omega_{ib}^b \approx 0$$

ECI Error Mechanization

Attitude

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i\Omega_{ib}^b = [\delta\dot{\vec{\psi}}^{\,i}_{ib}\times]\hat{C}_b^i + (\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\dot{\hat{C}}_b^i =$$

$$(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i(\hat{\Omega}_{ib}^b + \delta\Omega_{ib}^b) =$$

$$= \hat{C}_b^i\delta\Omega_{ib}^b + (\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i\hat{\Omega}_{ib}^b$$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b \ = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}\,_{ib}^i\times])\hat{C}_b^i\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}\,_{ib}^i\times])\hat{C}_b^i\Omega_{ib}^b = [\delta\dot{\vec{\psi}}\,_{ib}^i\times]\hat{C}_b^i + (\mathcal{I} + [\delta\vec{\psi}\,_{ib}^i\times])\dot{\hat{C}}_b^i =$$

$$(\mathcal{I} + [\delta\vec{\psi}\,_{ib}^i\times])\hat{C}_b^i(\hat{\Omega}_{ib}^b + \delta\Omega_{ib}^b) =$$

$$= \hat{C}_b^i\delta\Omega_{ib}^b + (\mathcal{I} + [\delta\vec{\psi}\,_{ib}^i\times])\hat{C}_b^i\hat{\Omega}_{ib}^b$$

$$[\delta\dot{\vec{\psi}}\,_{ib}^i\times] = \hat{C}_b^i\delta\Omega_{ib}^b\hat{C}_i^b = [\hat{C}_b^i\delta\vec{\omega}\,_{ib}^b\times] \tag{22}$$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}\,{}_{ib}^{i}\times])\hat{C}_b^i\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}\,{}_{ib}^{i}\times])\hat{C}_b^i\Omega_{ib}^b = [\delta\dot{\vec{\psi}}\,{}_{ib}^{i}\times]\hat{C}_b^i + (\mathcal{I} + [\delta\vec{\psi}\,{}_{ib}^{i}\times])\dot{\hat{C}}_b^i =$$

$$(\mathcal{I} + [\delta\vec{\psi}\,{}_{ib}^{i}\times])\hat{C}_b^i(\hat{\Omega}_{ib}^b + \delta\Omega_{ib}^b) =$$

$$= \hat{C}_b^i\delta\Omega_{ib}^b + (\mathcal{I} + [\delta\vec{\psi}\,{}_{ib}^{i}\times])\hat{C}_b^i\hat{\Omega}_{ib}^b$$

$$[\delta\dot{\vec{\psi}}\,{}_{ib}^{i}\times] = \hat{C}_b^i\delta\Omega_{ib}^b\hat{C}_i^b = [\hat{C}_b^i\delta\vec{\omega}\,{}_{ib}^{b}\times] \tag{22}$$

$$\delta\dot{\vec{\psi}}\,{}_{ib}^{i} = \hat{C}_b^i\delta\vec{\omega}\,{}_{ib}^{b} \tag{23}$$

$$\dot{\vec{v}}^{\,i}_{ib} = C^i_b \vec{f}^{\,b}_{ib} + \vec{\gamma}^{\,i}_{ib} \tag{24}$$

$$\dot{\vec{v}}^{\,i}_{ib} = C^i_b \vec{f}^{\,b}_{ib} + \vec{\gamma}^{\,i}_{ib} \tag{24}$$

$$\dot{\hat{\vec{v}}}^{\,i}_{ib} = \hat{C}^i_b \hat{\vec{f}}^{\,b}_{ib} + \hat{\vec{\gamma}}^{i}_{ib} \tag{25}$$

$$\hat{\vec{f}}^{\,b}_{ib} = \tilde{\vec{f}}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta\vec{f}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta_e\vec{f}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} - \delta\vec{f}^{\,b}_{ib}$$

$$\dot{\vec{v}}^{\,i}_{ib} = C^i_b\vec{f}^{\,b}_{ib} + \vec{\gamma}^{\,i}_{ib} \quad (24)$$

$$\dot{\hat{\vec{v}}}^{\,i}_{ib} = \hat{C}^i_b\hat{\vec{f}}^{\,b}_{ib} + \hat{\vec{\gamma}}^{i}_{ib} \quad (25)$$

$$\hat{\vec{f}}^{\,b}_{ib} = \tilde{\vec{f}}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta\vec{f}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta_e\vec{f}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} - \delta\vec{f}^{\,b}_{ib}$$

$$\dot{\vec{v}}^{\,i}_{ib} = C^i_b\vec{f}^{\,b}_{ib} + \vec{\gamma}^{\,i}_{ib} \tag{24}$$

$$\dot{\hat{\vec{v}}}^{\,i}_{ib} = \hat{C}^i_b\hat{\vec{f}}^{\,b}_{ib} + \hat{\vec{\gamma}}^{i}_{ib} = (\mathcal{I} - [\delta\vec{\psi}^{\,i}_{ib}\times])C^i_b(\vec{f}^{\,b}_{ib} + \Delta_e\vec{f}^{\,b}_{ib}) + \hat{\vec{\gamma}}^{i}_{ib} \tag{25}$$

$$\hat{\vec{f}}^{\,b}_{ib} = \tilde{\vec{f}}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta\vec{f}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta_e\vec{f}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} - \delta\vec{f}^{\,b}_{ib}$$

$$\dot{\vec{v}}^{\,i}_{ib} = C^i_b\vec{f}^{\,b}_{ib} + \vec{\gamma}^{\,i}_{ib} \tag{24}$$

$$\dot{\hat{\vec{v}}}^{\,i}_{ib} = \hat{C}^i_b\hat{\vec{f}}^{\,b}_{ib} + \hat{\vec{\gamma}}^i_{ib} = (\mathcal{I} - [\delta\vec{\psi}^{\,i}_{ib}\times])C^i_b(\vec{f}^{\,b}_{ib} + \Delta_e\vec{f}^{\,b}_{ib}) + \hat{\vec{\gamma}}^i_{ib} \tag{25}$$

$$\begin{aligned}\delta\dot{\vec{v}}^{\,i}_{ib} = \dot{\vec{v}}^{\,i}_{ib} - \dot{\hat{\vec{v}}}^i_{ib} &= [\delta\vec{\psi}^{\,i}_{ib}\times]C^i_b\vec{f}^{\,b}_{ib} + \hat{C}^i_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{\gamma}^{\,i}_{ib}\\ &= [\delta\vec{\psi}^{\,i}_{ib}\times]\hat{C}^i_b\hat{\vec{f}}^{\,b}_{ib} + \hat{C}^i_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{\gamma}^{\,i}_{ib}\end{aligned}$$

$$\hat{\vec{f}}\,^b_{ib} = \tilde{\vec{f}}\,^b_{ib} - \Delta\hat{\vec{f}}\,^b_{ib} = \vec{f}\,^b_{ib} + \Delta\vec{f}\,^b_{ib} - \Delta\hat{\vec{f}}\,^b_{ib} = \vec{f}\,^b_{ib} + \Delta_e\vec{f}\,^b_{ib} = \vec{f}\,^b_{ib} - \delta\vec{f}\,^b_{ib}$$

$$\dot{\vec{v}}\,^i_{ib} = C^i_b\vec{f}\,^b_{ib} + \vec{\gamma}\,^i_{ib} \tag{24}$$

$$\dot{\hat{\vec{v}}}\,^i_{ib} = \hat{C}^i_b\hat{\vec{f}}\,^b_{ib} + \hat{\vec{\gamma}}^i_{ib} = (\mathcal{I} - [\delta\vec{\psi}\,^i_{ib}\times])C^i_b(\vec{f}\,^b_{ib} + \Delta_e\vec{f}\,^b_{ib}) + \hat{\vec{\gamma}}^i_{ib} \tag{25}$$

$$\begin{aligned}\delta\dot{\vec{v}}\,^i_{ib} = \dot{\vec{v}}\,^i_{ib} - \dot{\hat{\vec{v}}}^i_{ib} &= [\delta\vec{\psi}\,^i_{ib}\times]C^i_b\vec{f}\,^b_{ib} + \hat{C}^i_b\delta\vec{f}\,^b_{ib} + \delta\vec{\gamma}\,^i_{ib} \\ &= [\delta\vec{\psi}\,^i_{ib}\times]\hat{C}^i_b\hat{\vec{f}}\,^b_{ib} + \hat{C}^i_b\delta\vec{f}\,^b_{ib} + \delta\vec{\gamma}\,^i_{ib}\end{aligned}$$

$$\delta\dot{\vec{v}}\,^i_{ib} = -[\hat{C}^i_b\hat{\vec{f}}\,^b_{ib}\times]\delta\vec{\psi}\,^i_{ib} + \hat{C}^i_b\delta\vec{f}\,^b_{ib} + \delta\vec{\gamma}\,^i_{ib} \tag{26}$$

$$\vec{\gamma}^{\,i}_{ib} \approx \frac{(r^e_{eS}(L_b))^2}{(r^e_{eS}(L_b) + h_b)^2} + \vec{\gamma}^{\,i}_0(L_b) \tag{27}$$

Assuming $h_b \ll r^e_{eS}$

$$\delta\vec{\gamma}^{\,i}_{ib} \approx -2\frac{(h_b - \hat{h}_b)}{r^e_{eS}(\hat{L}_b)} g_0(\hat{L}_b)\hat{\vec{u}}^i_D \tag{28}$$

$$\vec{\gamma}^{\,i}_{ib} \approx \frac{(r^e_{eS}(L_b))^2}{(r^e_{eS}(L_b) + h_b)^2} + \vec{\gamma}^{\,i}_{0}(L_b) \tag{27}$$

Assuming $h_b \ll r^e_{eS}$

$$\delta\vec{\gamma}^{\,i}_{ib} \approx -2\frac{(h_b - \hat{h}_b)}{r^e_{eS}(\hat{L}_b)} g_0(\hat{L}_b)\hat{\vec{u}}^i_D \tag{28}$$

Then converting from curvlinear coordinates to ECI

$$\delta\vec{\gamma}^{\,i}_{ib} \approx \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^{\,i}_{ib}}{|\hat{\vec{r}}^{\,i}_{ib}|^2} (\hat{\vec{r}}^{\,i}_{ib})^T \delta\vec{r}^{\,i}_{ib} \tag{29}$$

$$\dot{\vec{r}}^{\,i}_{ib} = \vec{v}^{\,i}_{ib} \tag{30}$$

$$\dot{\vec{r}}^{\,i}_{ib} = \vec{v}^{\,i}_{ib} \tag{30}$$

$$\delta\dot{\vec{r}}^{\,i}_{ib} = \delta\vec{v}^{\,i}_{ib} \tag{31}$$

Summary - in terms of $\delta\vec{f}_{ib}^{\,b}, \delta\vec{\omega}_{ib}^{\,b}$

$$\begin{pmatrix} \delta\dot{\vec{\psi}}_{ib}^{\,i} \\ \delta\dot{\vec{v}}_{ib}^{\,i} \\ \delta\dot{\vec{r}}_{ib}^{\,i} \end{pmatrix} = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ -[\hat{C}_b^i \hat{\vec{f}}_{ib}^{\,b}\times] & 0_{3\times3} & \frac{2g_0(\hat{L}_b)}{r_{eS}^e(\hat{L}_b)} \frac{\hat{\vec{r}}_{ib}^{\,i}}{|\hat{\vec{r}}_{ib}^{\,i}|^2} (\hat{\vec{r}}_{ib}^{\,i})^T \\ 0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{pmatrix} \delta\vec{\psi}_{ib}^{\,i} \\ \delta\vec{v}_{ib}^{\,i} \\ \delta\vec{r}_{ib}^{\,i} \end{pmatrix} + \begin{bmatrix} 0 & \hat{C}_b^i \\ \hat{C}_b^i & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \delta\vec{f}_{ib}^{\,b} \\ \delta\vec{\omega}_{ib}^{\,b} \end{pmatrix} \tag{32}$$

$$
\begin{pmatrix} \delta\dot{\vec{\psi}}^{\,i}_{ib} \\ \delta\dot{\vec{v}}^{\,i}_{ib} \\ \delta\dot{\vec{r}}^{\,i}_{ib} \end{pmatrix} = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ -[\hat{C}^i_b \hat{\vec{f}}^{\,b}_{ib}\times] & 0_{3\times3} & \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^{\,i}_{ib}}{|\hat{\vec{r}}^{\,i}_{ib}|^2} (\hat{\vec{r}}^{\,i}_{ib})^T \\ 0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{pmatrix} \delta\vec{\psi}^{\,i}_{ib} \\ \delta\vec{v}^{\,i}_{ib} \\ \delta\vec{r}^{\,i}_{ib} \end{pmatrix} + \begin{bmatrix} 0 & -\hat{C}^i_b \\ -\hat{C}^i_b & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \Delta_e \vec{f}^{\,b}_{ib} \\ \Delta_e \vec{\omega}^{\,b}_{ib} \end{pmatrix} \quad (33)
$$