Lecture

Navigation Mathematics: Kinematics (Coordinate Frame Transformation)

EE 570: Location and Navigation

Lecture Notes Update on February 16, 2016

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Coordinate Frame Transformation

- Determine the detailed kinematic relationships between the 4 major frames of interest
  - The Earth-Centered Inertial (ECI) coordinate frame (i-frame)
  - The Earth-Centered Earth-Fixed (ECEF) coordinate frame (e-frame)
  - The Local Navigation (Nav) coordinate frame (n-frame)
  - The Body coordinate frame (b-frame)

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ECI/ECEF

- Relationship between the ECI and ECEF frames
  - ECI & ECEF have co-located orgins

$$\vec{r}_{ie} = \dot{\vec{r}}_{ie} = \ddot{\vec{r}}_{ie} = 0$$

  - The x, y, and z axis of the ECI & ECEF frames are coincident at time t_0
  - The ECEF frame rotates about the common z-axis at a fixed rate (ω_{ie})
    * Ignoring minor speed variatins (precession & nutation) $\omega_{ie} = 72.921151467\mu$rad/sec (WGS84) which is $\approx 15°$/hr

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ECI/ECEF

- The angular velocity and acceleration are

$$\vec{\omega}^{\,i}_{ie} = \begin{bmatrix} 0 \\ 0 \\ \omega_{ie} \end{bmatrix} \qquad \dot{\vec{\omega}}^{\,i}_{ie} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- The angle of rotation is

$$\begin{aligned} \theta_{ie} &= \omega_{ie}(t - t_0) \\ &= \omega_{ie} t + \theta_{GMST} \end{aligned}$$

where GMST is the Greenwich mean sidereal time

- The orientation of frame $\{e\}$ *wrt* frame $\{i\}$ becomes

$$C_e^i = R_{(\vec{z},\theta_{ie})} = \begin{bmatrix} \cos\theta_{ie} & -\sin\theta_{ie} & 0 \\ \sin\theta_{ie} & \cos\theta_{ie} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: $\vec{\omega}\,^i_{ie} = \vec{\omega}\,^e_{ie}$

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ECEF/Nav

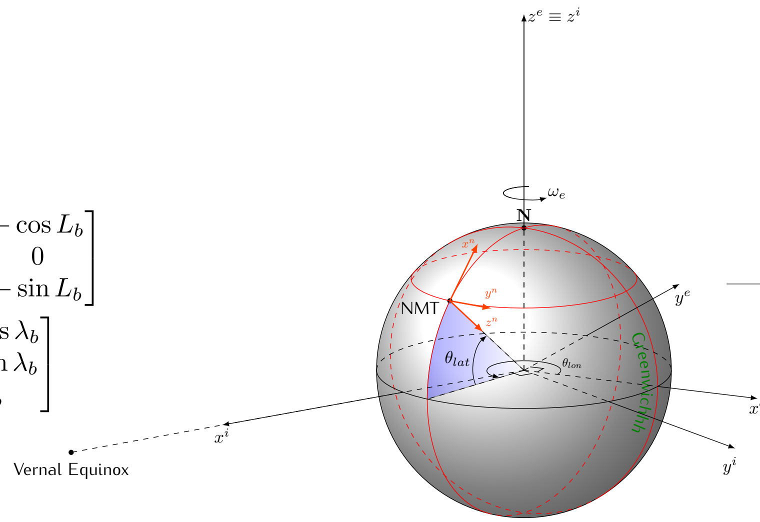
- Description of the navigation frame
  - Orientation of the n-frame *wrt* the e-frame

$$C_n^e = R_{(\vec{z},\lambda_b)} R_{(\vec{y},-L_b-90^\circ)}$$

$$= \begin{bmatrix} \cos\lambda_b & -\sin\lambda_b & 0 \\ \sin\lambda_b & \cos\lambda_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b & 0 & -\cos L_b \\ 0 & 1 & 0 \\ \cos L_b & 0 & -\sin L_b \end{bmatrix}$$

$$= \begin{bmatrix} -\sin L_b \cos\lambda_b & -\sin\lambda_b & -\cos L_b \cos\lambda_b \\ -\sin L_b \sin\lambda_b & \cos\lambda_b & -\cos L_b \sin\lambda_b \\ \cos L_b & 0 & -\sin L_b \end{bmatrix}$$

where geodetic Lat $= L_b$ and Geodetic Lon $= \lambda_b$



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ECEF/Nav

- Angular velocity of the n-frame *wrt* the e-frame resolved in the e-frame as a skew-symmetric matrix

$$\dot{C}_n^e = C_n^e \Omega_{en}^n = \Omega_{en}^e C_n^e$$

$$\Omega_{en}^e = \dot{C}_n^e \left[C_n^e\right]^T$$

$$= \begin{bmatrix} s_{L_b} s_{\lambda_b} \dot{\lambda}_b - c_{L_b} c_{\lambda_b} \dot{L}_b & -c_{\lambda_b} \dot{\lambda}_b & -c_{\lambda_b} s_{L_b} \dot{L}_b + c_{L_b} s_{\lambda_b} \dot{\lambda}_b \\ -c_{L_b} s_{\lambda_b} \dot{L}_b - c_{\lambda_b} s_{L_b} \dot{\lambda}_b & -s_{\lambda_b} \dot{\lambda}_b & s_{L_b} s_{\lambda_b} \dot{L}_b - c_{L_b} c_{\lambda_b} \dot{\lambda}_b \\ -s_{L_b} \dot{L}_b & 0 & -c_{L_b} \dot{L}_b \end{bmatrix} \left[C_n^e\right]^T$$

$$= \begin{bmatrix} 0 & -\dot{\lambda}_b & -\dot{L}_b \cos(\lambda_b) \\ \dot{\lambda}_b & 0 & -\dot{L}_b \sin(\lambda_b) \\ \dot{L}_b \cos(\lambda_b) & \dot{L}_b \sin(\lambda_b) & 0 \end{bmatrix}$$

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ECEF/Nav

- The angular velocity vector

$$\vec{\omega}\,^e_{en} = \begin{bmatrix} \sin(\lambda)\dot{L}_b \\ -\cos(\lambda)\dot{L}_b \\ \dot{\lambda}_b \end{bmatrix} \qquad \vec{\omega}\,^n_{en} = \left[C_n^e\right]^T \omega_{en}^e = \begin{bmatrix} \cos(\lambda)\dot{\lambda}_b \\ -\dot{L}_b \\ -\sin(L_b)\dot{\lambda}_b \end{bmatrix}$$

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ECI/ECEF/Nav

- Hence the orientation of the n-frame *wrt* the i-frame becomes

$$C_n^i = C_e^i C_n^e = \begin{bmatrix} c_{\theta_{ie}} & -s_{\theta_{ie}} & 0 \\ s_{\theta_{ie}} & c_{\theta_{ie}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_{L_b} c_{\lambda_b} & -s_{\lambda_b} & -c_{L_b} c_{\lambda_b} \\ -s_{L_b} s_{\lambda_b} & c_{\lambda_b} & -c_{L_b} s_{\lambda_b} \\ c_{L_b} & 0 & -s_{L_b} \end{bmatrix}$$

$$= \begin{bmatrix} -\sin(L_b)\cos(\theta_{ie}+\lambda_b) & -\sin(\theta_{ie}+\lambda_b) & -\cos(L_b)\cos(\theta_{ie}+\lambda_b) \\ -\sin(L_b)\sin(\theta_{ie}+\lambda_b) & \cos(\theta_{ie}+\lambda_b) & -\cos(L_b)\sin(\theta_{ie}+\lambda_b) \\ \cos(L_b) & 0 & -\sin(L_b) \end{bmatrix}$$

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ECI/ECEF/Nav

- The angular velocity of the n-frame *wrt* the i-frame resolved in the i-frame is

$$\vec{\omega}^{\,i}_{in} = \vec{\omega}^{\,i}_{ie} + C_e^i \vec{\omega}^{\,e}_{en}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \omega_{ie} \end{bmatrix} + \begin{bmatrix} \cos(\theta_{ie}) & -\sin(\theta_{ie}) & 0 \\ \sin(\theta_{ie}) & \cos(\theta_{ie}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(\lambda_b)\dot{L}_b \\ -\cos(\lambda_b)\dot{L}_b \\ \dot{\lambda}_b \end{bmatrix}$$

$$= \begin{bmatrix} \sin(\theta_{ie}+\lambda_b)\dot{L}_b \\ -\cos(\theta_{ie}+\lambda_b)\dot{L}_b \\ \omega_{ie}+\dot{\lambda}_b \end{bmatrix}$$

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ECEF/Nav

- The vector from the origin of the e-frame to the n-frame origin resolved in the e-frame (from the last lecture)

$$\vec{r}^{\,e}_{eb} = \begin{bmatrix} (R_E+h_b)\cos(L_b)\cos(\lambda_b) \\ (R_E+h_b)\cos(L_b)\sin(\lambda_b) \\ (R_E(1-e^2)+h_b)\sin(L_b) \end{bmatrix} = \vec{r}^{\,e}_{en}$$

> Origins of the n-frame and the b-frame are the same

- The velocity of the n-frame *wrt* the e-frame resolved in the e-frame

$$\vec{v}^{\,e}_{en} = \frac{d}{dt}\vec{r}^{\,e}_{en} = \frac{\partial \vec{r}^{\,e}_{en}}{\partial L_b}\dot{L}_b + \frac{\partial \vec{r}^{\,e}_{en}}{\partial \lambda_b}\dot{\lambda}_b + \frac{\partial \vec{r}^{\,e}_{en}}{\partial h_b}\dot{h}_b$$

$$= \begin{bmatrix} -\sin(L_b)\cos(\lambda_b) & -\sin(\lambda_b) & -\cos(L_b)\cos(\lambda_b) \\ -\sin(L_b)\sin(\lambda_b) & \cos(\lambda_b) & -\cos(L_b)\sin(\lambda_b) \\ \cos(L_b) & 0 & -\sin(L_b) \end{bmatrix} \begin{bmatrix} (R_N+h_b)\dot{L}_b \\ \cos(L_b)(R_E+h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix}$$

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ECEF/Nav

- Recalling the form of C_n^e suggests that

$$\vec{v}^{\,e}_{en} = C_n^e \begin{bmatrix} (R_N+h_b)\dot{L}_b \\ \cos(L_b)(R_E+h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix} = C_n^e \vec{v}^{\,n}_{en}$$

- and hence,

$$\vec{v}^{\,n}_{en} = \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix}$$

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ECEF/Nav

- Restating $\vec{v}^{\,n}_{en}$ as

$$\vec{v}^{\,n}_{en} = \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix} = \begin{bmatrix} v^n_{en,N} \\ v^n_{en,E} \\ v^n_{en,D} \end{bmatrix}$$

- and recalling that

$$\vec{\omega}^{\,n}_{en} = \begin{bmatrix} \cos(L_b)\dot{\lambda}_b \\ -\dot{L}_b \\ -\sin(L_b)\dot{\lambda}_b \end{bmatrix}$$

- suggests that

$$\vec{\omega}^{\,n}_{en} = \begin{bmatrix} \vec{v}^{\,n}_{en,E}/(R_E + h_b) \\ -\vec{v}^{\,n}_{en,N}/(R_N + h_b) \\ -\tan(L_b)\vec{v}^{\,n}_{en,E}/(R_E + h_b) \end{bmatrix}$$

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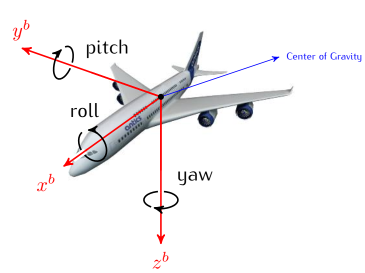
Body Frame

- Description *wrt* the body frame
  - Orientation of the b-frame *wrt* the n-frame in terms of relative yaw (ψ), pitch (θ), then roll (ϕ) angles

$$C^n_b = R_{(\vec{z},\psi)}R_{(\vec{y},\theta)}R_{(\vec{x},\phi)} = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$



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Body Frame

- The angular velocity of the b-frame *wrt* the i-frame resolved/coordinatized in the i-frame

$$\begin{aligned}\vec{\omega}^{\,i}_{ib} &= \vec{\omega}^{\,i}_{in} + C^i_n \vec{\omega}^{\,n}_{nb} \\ &= \vec{\omega}^{\,i}_{ie} + C^i_e \vec{\omega}^{\,e}_{en} + C^i_n \vec{\omega}^{\,n}_{nb}\end{aligned}$$

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Body Frame

- Position vectors to the origin of the body frame
  - The origins of the body and Nav frames are co-incident

$$\vec{r}_{nb} = \vec{0}$$

  - The origins of the ECI and ECEF frames are co-incident

$$\vec{r}_{eb} = \vec{r}_{ib} = \vec{r}_{en} = \vec{r}_{in}$$

  - Velocity of the b-frame *wrt* the i-frame resolved in the i-frame
    * A moving point in a rotation frame

$$\begin{aligned}\vec{v}^{\,i}_{ib} = \frac{d}{dt}\vec{r}^{\,i}_{ib} &= \frac{d}{dt} C^i_e \vec{r}^{\,e}_{eb} \\ &= C^i_e \Omega^e_{ie} \vec{r}^{\,e}_{eb} + C^i_e \vec{v}^{\,e}_{eb} \\ &= C^i_e \left(\Omega^e_{ie} \vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb} \right)\end{aligned}$$

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Body Frame

- Acceleration of the b-frame *wrt* the i-frame resolved in the i-frame
  - A moving point in a rotation frame

$$\begin{aligned}\vec{a}^{\,i}_{ib} = \frac{d}{dt}\vec{v}^{\,i}_{ib} &= \frac{d}{dt}\left(C^i_e \left(\Omega^e_{ie} \vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb} \right)\right) \\ &= \dot{C}^i_e \left(\Omega^e_{ie} \vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb} \right) + C^i_e \begin{pmatrix} \dot{\omega} = 0 \\ \dot{\Omega}^e_{ie}\, \vec{r}^{\,e}_{eb} + \Omega^e_{ie} \dot{\vec{r}}^{\,e}_{eb} + \dot{\vec{v}}^{\,e}_{eb} \end{pmatrix} \\ &= C^i_e \Omega^e_{ie} \left(\Omega^e_{ie} \vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb} \right) + C^i_e \left(\Omega^e_{ie} \vec{v}^{\,e}_{eb} + \vec{a}^{\,e}_{eb} \right) \\ &= C^i_e \left(\Omega^e_{ie} \Omega^e_{ie} \vec{r}^{\,e}_{eb} + 2\Omega^e_{ie} \vec{v}^{\,e}_{eb} + \vec{a}^{\,e}_{eb} \right)\end{aligned}$$

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