

EE 570: Location and Navigation

Navigation Mathematics: Kinematics (Coordinate Frame Transformation)

Aly El-Osery Kevin Wedeward

Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA

In Collaboration with
Stephen Bruder
Electrical and Computer Engineering Department
Embry-Riddle Aeronautical University
Prescott, Arizona, USA

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- Determine the detailed kinematic relationships between the 4 major frames of interest
 - The Earth-Centered Inertial (ECI) coordinate frame (i -frame)
 - The Earth-Centered Earth-Fixed (ECEF) coordinate frame (e -frame)
 - The Local Navigation (Nav) coordinate frame (n -frame)
 - The Body coordinate frame (b -frame)

- Relationship between the ECI and ECEF frames
 - ECI & ECEF have co-located origins

$$\vec{r}_{ie} = \dot{\vec{r}}_{ie} = \ddot{\vec{r}}_{ie} = 0$$

- The x , y , and z axis of the ECI & ECEF frames are coincident at time t_0
- The ECEF frame rotates about the common z -axis at a fixed rate (ω_{ie})
 - Ignoring minor speed variations (precession & nutation)
 $\omega_{ie} = 72.921151467 \mu\text{rad}/\text{sec}$ (WGS84) which is $\approx 15^\circ/\text{hr}$

- The angular velocity and acceleration are

$$\vec{\omega}_{ie}^i = \begin{bmatrix} 0 \\ 0 \\ \omega_{ie} \end{bmatrix} \quad \dot{\vec{\omega}}_{ie}^i = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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- The angle of rotation is

$$\begin{aligned} \theta_{ie} &= \omega_{ie}(t - t_0) \\ &= \omega_{ie}t + \theta_{GMST} \end{aligned}$$

where **GMST** is the Greenwich mean sidereal time

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- The orientation of frame $\{e\}$ wrt frame $\{i\}$ becomes

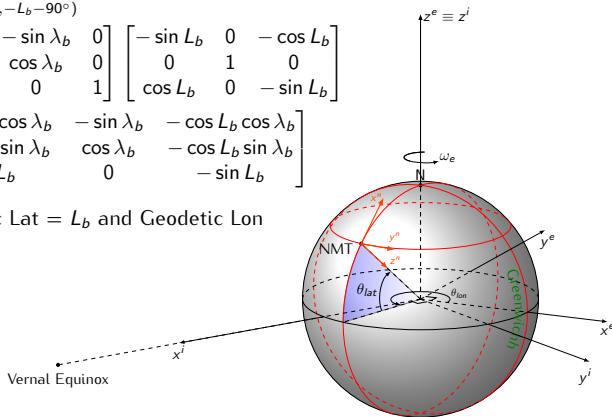
$$C_e^i = R_{(\vec{z}, \theta_{ie})} = \begin{bmatrix} \cos \theta_{ie} & -\sin \theta_{ie} & 0 \\ \sin \theta_{ie} & \cos \theta_{ie} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: $\vec{\omega}_{ie}^i = \vec{\omega}_{ie}^e$

- Description of the navigation frame
 - Orientation of the n -frame wrt the e -frame

$$\begin{aligned}
 C_n^e &= R_{(z, \lambda_b)} R_{(y, -L_b - 90^\circ)} \\
 &= \begin{bmatrix} \cos \lambda_b & -\sin \lambda_b & 0 \\ \sin \lambda_b & \cos \lambda_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b & 0 & -\cos L_b \\ 0 & 1 & 0 \\ \cos L_b & 0 & -\sin L_b \end{bmatrix} \\
 &= \begin{bmatrix} -\sin L_b \cos \lambda_b & -\sin \lambda_b & -\cos L_b \cos \lambda_b \\ -\sin L_b \sin \lambda_b & \cos \lambda_b & -\cos L_b \sin \lambda_b \\ \cos L_b & 0 & -\sin L_b \end{bmatrix}
 \end{aligned}$$

where geodetic Lat = L_b and Geodetic Lon = λ_b



- Angular velocity of the n -frame *wrt* the e -frame resolved in the e -frame as a skew-symmetric matrix

- Angular velocity of the n -frame wrt the e -frame resolved in the e -frame as a skew-symmetric matrix

$$\begin{aligned}
 \Omega_{en}^e &= \dot{C}_n^e [C_n^e]^T && \dot{C}_n^e = C_n^e \Omega_{en}^n = \Omega_{en}^e C_n^e \\
 &= \begin{bmatrix} s_{L_b} s_{\lambda_b} \dot{\lambda}_b - c_{L_b} c_{\lambda_b} \dot{L}_b & -c_{\lambda_b} \dot{\lambda}_b & -c_{\lambda_b} s_{L_b} \dot{L}_b + c_{L_b} s_{\lambda_b} \dot{\lambda}_b \\ -c_{L_b} s_{\lambda_b} \dot{L}_b - c_{\lambda_b} s_{L_b} \dot{\lambda}_b & -s_{\lambda_b} \dot{\lambda}_b & s_{L_b} s_{\lambda_b} \dot{L}_b - c_{L_b} c_{\lambda_b} \dot{\lambda}_b \\ -s_{L_b} \dot{L}_b & 0 & -c_{L_b} \dot{L}_b \end{bmatrix} [C_n^e]^T \\
 &= \begin{bmatrix} 0 & -\dot{\lambda}_b & -\dot{L}_b \cos(\lambda_b) \\ \dot{\lambda}_b & 0 & -\dot{L}_b \sin(\lambda_b) \\ \dot{L}_b \cos(\lambda_b) & \dot{L}_b \sin(\lambda_b) & 0 \end{bmatrix}
 \end{aligned}$$

- The angular velocity vector

$$\vec{\omega}_{en}^e = \begin{bmatrix} \sin(\lambda)\dot{L}_b \\ -\cos(\lambda)\dot{L}_b \\ \dot{\lambda}_b \end{bmatrix} \quad \vec{\omega}_{en}^n = [C_n^e]^T \omega_{en}^e = \begin{bmatrix} \cos(\lambda)\dot{\lambda}_b \\ -\dot{L}_b \\ -\sin(L_b)\dot{\lambda}_b \end{bmatrix}$$

- Hence the orientation of the n -frame *wrt* the i -frame becomes

- Hence the orientation of the n -frame wrt the i -frame becomes

$$\begin{aligned}
 C_n^i &= C_e^i C_n^e = \begin{bmatrix} c_{\theta_{ie}} & -s_{\theta_{ie}} & 0 \\ s_{\theta_{ie}} & c_{\theta_{ie}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_{L_b} c_{\lambda_b} & -s_{\lambda_b} & -c_{L_b} c_{\lambda_b} \\ -s_{L_b} s_{\lambda_b} & c_{\lambda_b} & -c_{L_b} s_{\lambda_b} \\ c_{L_b} & 0 & -s_{L_b} \end{bmatrix} \\
 &= \begin{bmatrix} -\sin(L_b) \cos(\theta_{ie} + \lambda_b) & -\sin(\theta_{ie} + \lambda_b) & -\cos(L_b) \cos(\theta_{ie} + \lambda_b) \\ -\sin(L_b) \sin(\theta_{ie} + \lambda_b) & \cos(\theta_{ie} + \lambda_b) & -\cos(L_b) \sin(\theta_{ie} + \lambda_b) \\ \cos(L_b) & 0 & -\sin(L_b) \end{bmatrix}
 \end{aligned}$$

- The angular velocity of the n -frame wrt the i -frame resolved in the i -frame is

- The angular velocity of the n -frame wrt the i -frame resolved in the i -frame is

$$\begin{aligned}
 \vec{\omega}_{in}^i &= \vec{\omega}_{ie}^i + C_e^i \vec{\omega}_{en}^e \\
 &= \begin{bmatrix} 0 \\ 0 \\ \omega_{ie} \end{bmatrix} + \begin{bmatrix} \cos(\theta_{ie}) & -\sin(\theta_{ie}) & 0 \\ \sin(\theta_{ie}) & \cos(\theta_{ie}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(\lambda_b) \dot{L}_b \\ -\cos(\lambda_b) \dot{L}_b \\ \dot{\lambda}_b \end{bmatrix} \\
 &= \begin{bmatrix} \sin(\theta_{ie} + \lambda_b) \dot{L}_b \\ -\cos(\theta_{ie} + \lambda_b) \dot{L}_b \\ \omega_{ie} + \dot{\lambda}_b \end{bmatrix}
 \end{aligned}$$

- The vector from the origin of the e -frame to the n -frame origin resolved in the e -frame (from the last lecture)

$$\vec{r}_{eb}^e = \begin{bmatrix} (R_E + h_b) \cos(L_b) \cos(\lambda_b) \\ (R_E + h_b) \cos(L_b) \sin(\lambda_b) \\ (R_E(1 - e^2) + h_b) \sin(L_b) \end{bmatrix} = \vec{r}_{en}^e$$

Origins of the n -frame and the b -frame are the same

- The velocity of the n -frame wrt the e -frame resolved in the e -frame

$$\begin{aligned} \vec{v}_{en}^e &= \frac{d}{dt} \vec{r}_{en}^e = \frac{\partial \vec{r}_{en}^e}{\partial L_b} \dot{L}_b + \frac{\partial \vec{r}_{en}^e}{\partial \lambda_b} \dot{\lambda}_b + \frac{\partial \vec{r}_{en}^e}{\partial h_b} \dot{h}_b \\ &= \begin{bmatrix} -\sin(L_b) \cos(\lambda_b) & -\sin(\lambda_b) & -\cos(L_b) \cos(\lambda_b) \\ -\sin(L_b) \sin(\lambda_b) & \cos(\lambda_b) & -\cos(L_b) \sin(\lambda_b) \\ \cos(L_b) & 0 & -\sin(L_b) \end{bmatrix} \begin{bmatrix} (R_E + h_b) \dot{L}_b \\ \cos(L_b)(R_E + h_b) \dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix} \end{aligned}$$

- Recalling the form of C_n^e suggests that

$$\vec{v}_{en}^e = C_n^e \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix} = C_n^e \vec{v}_{en}^n$$

- and hence,

$$\vec{v}_{en}^n = \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix}$$

- Restating \vec{v}_{en}^n as

$$\vec{v}_{en}^n = \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix} = \begin{bmatrix} v_{en,N}^n \\ v_{en,E}^n \\ v_{en,D}^n \end{bmatrix}$$

- and recalling that

$$\vec{\omega}_{en}^n = \begin{bmatrix} \cos(L_b)\dot{\lambda}_b \\ -\dot{L}_b \\ -\sin(L_b)\dot{\lambda}_b \end{bmatrix}$$

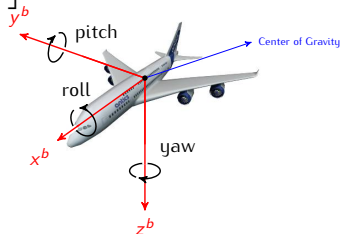
- suggests that

$$\vec{\omega}_{en}^n = \begin{bmatrix} \vec{v}_{en,E}^n / (R_E + h_b) \\ -\vec{v}_{en,N}^n / (R_N + h_b) \\ -\tan(L_b)\vec{v}_{en,E}^n / (R_E + h_b) \end{bmatrix}$$

- Description *wrt* the body frame
 - Orientation of the b -frame *wrt* the n -frame in terms of relative yaw (ψ), pitch (θ), then roll (ϕ) angles

$$C_b^n = R_{(\bar{z}, \psi)} R_{(\bar{y}, \theta)} R_{(\bar{x}, \phi)} = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$



- The angular velocity of the b -frame *wrt* the i -frame resolved/coordinated in the i -frame

- The angular velocity of the b -frame *wrt* the i -frame resolved/coordinated in the i -frame

$$\begin{aligned}\vec{\omega}_{ib}^i &= \vec{\omega}_{in}^i + C_n^i \vec{\omega}_{nb}^n \\ &= \vec{\omega}_{ie}^i + C_e^i \vec{\omega}_{en}^e + C_n^i \vec{\omega}_{nb}^n\end{aligned}$$

- Position vectors to the origin of the body frame
 - The origins of the body and Nav frames are co-incident

$$\vec{r}_{nb} = \vec{0}$$

- The origins of the ECI and ECEF frames are co-incident

$$\vec{r}_{eb} = \vec{r}_{ib} = \vec{r}_{en} = \vec{r}_{in}$$

- Velocity of the b -frame *wrt* the i -frame resolved in the i -frame
 - A moving point in a rotation frame

$$\begin{aligned}
 \vec{v}_{ib}^i &= \frac{d}{dt} \vec{r}_{ib}^i = \frac{d}{dt} C_e^i \vec{r}_{eb}^e \\
 &= C_e^i \Omega_{ie}^e \vec{r}_{eb}^e + C_e^i \vec{v}_{eb}^e \\
 &= C_e^i (\Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e)
 \end{aligned}$$

- Acceleration of the b -frame wrt the i -frame resolved in the i -frame
 - A moving point in a rotation frame

$$\begin{aligned}
 \vec{a}_{ib}^i &= \frac{d}{dt} \vec{v}_{ib}^i = \frac{d}{dt} (C_e^i (\Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e)) \\
 &= \dot{C}_e^i (\Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e) + C_e^i \left(\overset{\dot{\omega} = 0}{\cancel{\dot{\Omega}_{ie}^e}} \vec{r}_{eb}^e + \Omega_{ie}^e \dot{\vec{r}}_{eb}^e + \dot{\vec{v}}_{eb}^e \right) \\
 &= C_e^i \Omega_{ie}^e (\Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e) + C_e^i (\Omega_{ie}^e \dot{\vec{v}}_{eb}^e + \ddot{\vec{a}}_{eb}^e) \\
 &= C_e^i (\Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e + 2\Omega_{ie}^e \dot{\vec{v}}_{eb}^e + \ddot{\vec{a}}_{eb}^e)
 \end{aligned}$$