Lecture

Navigation Mathematics: Rotation Matrices, Part II

EE 570: Location and Navigation

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Lecture Topics

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1 Review

Review

Rotation matrix, C_2^1

- describes orientation of frame 2 with respect to frame 1
- is of size 3×3
- is constructed via $\left[x_2^1,\ y_2^1,\ z_2^1\right] = \begin{bmatrix} x_2 \cdot x_1, & y_2 \cdot x_1, & z_2 \cdot x_1 \\ x_2 \cdot y_1, & y_2 \cdot y_1, & z_2 \cdot y_1 \\ x_2 \cdot z_1, & y_2 \cdot z_1, & z_2 \cdot z_1 \end{bmatrix}$
- has inverse $[C_2^1]^{-1} = [C_2^1]^T = C_1^2$
- is of the form $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{z,\theta}$ for the basic (elementary) rotation about the $z-$axis by angle θ

similarly,

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad R_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

- recoordinatizes vector $\vec{v}^{\,2}$ in frame 1 via $\vec{v}^{\,1} = C_2^1 \vec{v}^{\,2}$

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2 Parameterizations of Rotations

Parameterizations of Rotations

Many approaches to parameterize orientation

1. Rotation matrices use $3 \times 3 = 9$ parameters
   - these 9 parameters are not independent
   - 3 constraints due to columns being orthogonal
   - 3 constraints due to columns being unit vectors

   $\Rightarrow 3$ free variables exist $\Rightarrow$ need only 3 parameters to describe orientation
2. Examples of $3-$parameter descriptions:
   - fixed-axis rotations (e.g., Roll-Pitch-Yaw/ZYX)
   - relative-axis (Euler) rotations (e.g., ZYZ, ZYX)
   - angle and axis
3. Quaternions use 4 parameters

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3 Fixed versus Relative Rotations

Fixed versus Relative Rotations

When one wants to rotate a coordinate frame about an axis, that axis can be in a fixed-frame or relative-frame.

1. Fixed-axis rotation - rotation performed about $x-$, $y-$, or $z-$axis of initial (and fixed) coordinate frame
2. Relative-axis rotation - rotation performed about $x-$, $y-$, or $z-$axis of current (and relative) coordinate frame
   - sometimes referred to as Euler rotations

Resulting orientation is quite different!

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Example Sequence of Rotations

Example sequence of three consecutive rotations to compare fixed versus relative.

- **Step 1:** Rotate about the z-axis by ψ
- **Step 2:** Rotate about the y-axis by θ
- **Step 3:** Rotate about the x-axis by ϕ

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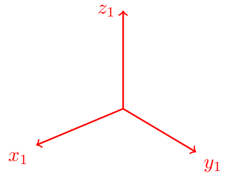
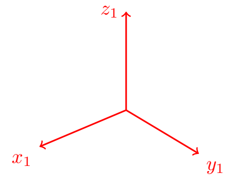
Example Sequence of Rotations

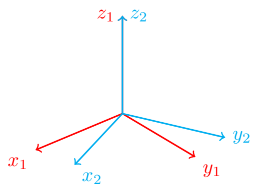
Relative Axis Rotation



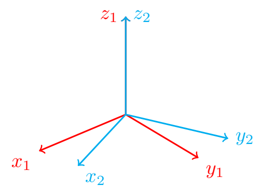
Fixed Axis Rotation



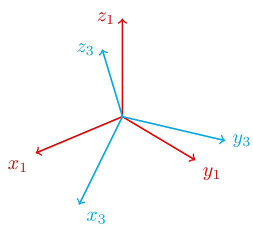
Relative Axis Rotation
Rotate about z_1



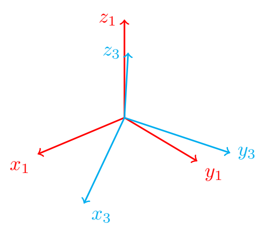
Fixed Axis Rotation
Rotate about z_1



Relative Axis Rotation
Rotate about y_2



Fixed Axis Rotation
Rotate about y_1



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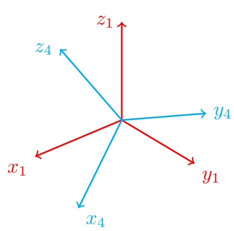
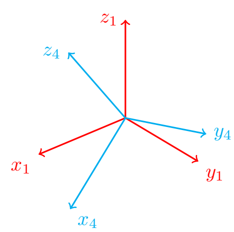
Relative Axis Rotation
Rotate about x_3



Fixed Axis Rotation
Rotate about x_1



4 Composition of Relative-axis Rotations

Composition of Relative-axis Rotations

Construct rotation matrix that represents composition of relative-axis rotations.

- Start with last rotation $C_4^3 = [x_4^3, y_4^3, z_4^3] = R_{x,\phi}$, and recall columns are vectors
- To re-coordinatize vectors x_4^3, y_4^3, z_4^3 in frame 2, multiply each by $C_3^2 = R_{y,\theta}$.

  In matrix form, $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = [x_4^2, y_4^2, z_4^2] = C_4^2$

  where it is noted that $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = C_3^2 [x_4^3, y_4^3, z_4^3] = C_3^2 C_4^3 = C_4^2$

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Composition of Relative-axis Rotations

- To re-coordinatize vectors x_4^2, y_4^2, z_4^2 in frame 1, multiply each by $C_2^1 = R_{z,\psi}$

  $\Rightarrow [C_2^1 x_4^2, C_2^1 y_4^2, C_2^1 z_4^2] = C_2^1[x_4^2, y_4^2, z_4^2] = C_2^1 C_4^2 = C_2^1 C_3^2 C_4^3 = C_4^1$

- Combined sequence of relative-rotations yields

  $C_4^1 = C_2^1 C_3^2 C_4^3 = \underbrace{R_{z,\psi}}_{1st}\underbrace{R_{y,\theta}}_{2nd}\underbrace{R_{x,\phi}}_{3rd}$

- Note order is left to right!

- Additional relative-rotations represented by right (post) matrix multiplies.

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Rotation Matrix from Relative ZYX

For the relative-axis rotations Z(ψ), Y(θ), X(ϕ)

$$\begin{aligned} C_4^1 &= C_2^1 C_3^2 C_4^3 \\ &= R_{z,\psi} R_{y,\theta} R_{x,\phi} \\ &= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \end{aligned}$$

where the notation $c_\beta = \cos(\beta)$ and $s_\beta = \sin(\beta)$ are introduced .10

5 Composition of Fixed-axis Rotations

Composition of Fixed-axis Rotations

- Development of equivalent rotation matrix for sequence of fixed-axis rotations will make use of rotation matrix's ability to rotate a vector.
- A vector $\vec{p}$ can be rotated into a new vector via $R\vec{p}$, both in the same coordinate frame.
- We'll once again consider the sequence Z(ψ), Y(θ), X(ϕ) aka Yaw-Pitch-Roll, but this time about fixed-axes.

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Composition of Fixed-axis Rotations

- First $z-$axis rotation rotates frame {1}'s basis vectors to become frame {2}'s basis vectors $[\vec{x}\,_2^1, \vec{y}\,_2^1, \vec{z}\,_2^1] = R_{z,\psi}[\vec{x}\,_1^1, \vec{y}\,_1^1, \vec{z}\,_1^1] = R_{z,\psi}$
- Second $y-$axis rotation rotates frame {2}'s basis vectors to become frame {3}'s basis vectors $[\vec{x}\,_3^1, \vec{y}\,_3^1, \vec{z}\,_3^1] = R_{y,\theta}[\vec{x}\,_2^1, \vec{y}\,_2^1, \vec{z}\,_2^1] = R_{y,\theta} R_{z,\psi}$
- Third $x-$axis rotation rotates frame {3}'s basis vectors to become frame {4}'s basis vector $[\vec{x}\,_4^1, \vec{y}\,_4^1, \vec{z}\,_4^1] = R_{x,\phi}[\vec{x}\,_3^1, \vec{y}\,_3^1, \vec{z}\,_3^1] = R_{x,\phi} R_{y,\theta} R_{z,\psi}$
  $\Rightarrow C_4^1 = \underbrace{R_{x,\phi}}_{3rd}\underbrace{R_{y,\theta}}_{2nd}\underbrace{R_{z,\psi}}_{1st}$
- Note order is right to left!
- Additional fixed-rotations represented by left (pre) matrix multiplies.

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Composition of Fixed-axis Rotations

For the fixed-axis rotations Z(ψ), Y(θ), X(ϕ)

$$\begin{aligned} C_4^1 &= R_{x,\phi} R_{y,\theta} R_{z,\psi} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\psi s_\theta s_\phi + c_\phi s_\psi & c_\phi c_\psi - s_\theta s_\phi s_\psi & -c_\theta s_\phi \\ s_\phi s_\psi - c_\phi c_\psi s_\theta & c_\psi s_\phi + c_\phi s_\theta s_\psi & c_\theta c_\phi \end{bmatrix} \end{aligned}$$

which is quite different than the result for the same sequence of relative-axis rotations .13

6 Example

Example

Find the rotation matrix representing the orientation of the coordinate frame that results from the following sequence of rotations.

1. Rotate about fixed $x-$axis by ϕ
2. Rotate about fixed $z-$axis by θ
3. Rotate about current $x-$axis by ψ
4. Rotate about current $z-$axis by α
5. Rotate about fixed $y-$axis by β
6. Rotate about current $y-$axis by γ

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7 Summary

Fixed vs Relative Rotations

- Fixed-axis Rotations
  - Multiply on the *LEFT*
  - $C_{final} = R_n \dots R_2 R_1$

Fixed-axis Rotation
$C_{resultant} = R_{fixed} C_{original}$

- Relative-axis (Euler) Rotations
  - Multiply on the *RIGHT*
  - $C_{final} = R_1 R_2 \dots R_n$

Relative-axis Rotation
$C_{resultant} = C_{original} R_{relative}$

Two types of rotations can be composed noting order of multiplication

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The End

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