Kevin Wedeward  Aly El-Osery

Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA

In Collaboration with
Stephen Bruder
Electrical and Computer Engineering Department
Embry-Riddle Aeronautical University
Prescott, Arizona, USA

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Lecture Topics

1. Review
2. Parameterizations of Rotations
3. Fixed versus Relative Rotations
4. Composition of Relative-axis Rotations
5. Composition of Fixed-axis Rotations
6. Example
7. Summary
Rotation matrix, $C_2^1$
- describes orientation of
Rotation matrix, $C^1_2$

- describes orientation of frame 2 with respect to frame 1
Rotation matrix, $C_2^1$

- describes orientation of frame 2 with respect to frame 1
- is of size
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- is of size $3 \times 3$
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- is of size $3 \times 3$
- is constructed via

$$\begin{bmatrix}
  x_2 \cdot x_1, \\
y_2 \cdot x_1, \\
z_2 \cdot x_1
\end{bmatrix},
\begin{bmatrix}
  x_2 \cdot y_1, \\
y_2 \cdot y_1, \\
z_2 \cdot y_1
\end{bmatrix},
\begin{bmatrix}
  x_2 \cdot z_1, \\
y_2 \cdot z_1, \\
z_2 \cdot z_1
\end{bmatrix}$$

has inverse $C_1^2 = C_2^1 \cdot C_1^2$ is of the form

$$\begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos(\theta) & -\sin(\theta) \\
  0 & \sin(\theta) & \cos(\theta)
\end{bmatrix}$$

similarly, $R_x,\theta = \begin{bmatrix} 1 & 0 & 0 \\
0 & \cos(\theta) & -\sin(\theta) \\
0 & \sin(\theta) & \cos(\theta)\end{bmatrix}$,

$R_y,\theta = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\
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recoordinatizes vector $\vec{v}_2$ in frame 1 via $\vec{v}_1 = C_1^2 \vec{v}_2$
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has inverse $[C_2^1]^{-1} = [C_2^1]^T = C_1^2$ is of the form $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{z,\theta}$ for the basic (elementary) rotation about the $z$-axis by angle $\theta$

similarly, $R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$,

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similarly,

$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$, $R_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$
Rotation matrix, $C^1_2$

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is of the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{z,\theta}$$

for the basic (elementary) rotation about the $z$–axis by angle $\theta$

similarly,

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}, \quad R_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

recoordinatizes vector $\vec{v}^2$ in frame 1 via $\vec{v}^1 =$
Review

Rotation matrix, $C^1_2$

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- similarly, $R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$, $R_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$

- recoordinates vector $\vec{v}^2$ in frame 1 via $\vec{v}^1 = C^1_2 \vec{v}^2$
Many approaches to parameterize orientation

- Rotation matrices use $3 \times 3 = 9$ parameters

Examples of 3-parameter descriptions:
- Fixed-axis rotations (e.g., Roll-Pitch-Yaw/ZYX)
- Relative-axis (Euler) rotations (e.g., ZYZ, ZYX)
- Angle and axis
Parameterizations of Rotations

Many approaches to parameterize orientation

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  - 3 constraints due to columns being unit vectors
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   - angle and axis
Parameterizations of Rotations

Many approaches to parameterize orientation

1. Rotation matrices use $3 \times 3 = 9$ parameters
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   - 3 constraints due to columns being orthogonal
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   \[ \Rightarrow 3 \text{ free variables exist} \Rightarrow \text{need only 3 parameters to describe orientation} \]

2. Examples of 3–parameter descriptions:
   - fixed-axis rotations (e.g., Roll-Pitch-Yaw/ZYX)
   - relative-axis (Euler) rotations (e.g., ZYZ, ZYX)
   - angle and axis

3. Quaternions use 4 parameters
When one wants to rotate a coordinate frame about an axis, that axis can be in a fixed-frame or relative-frame.

1. **Fixed-axis rotation** - rotation performed about $x-$, $y-$, or $z-$axis of initial (and fixed) coordinate frame
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1. **Fixed-axis rotation** - rotation performed about $x-$, $y-$, or $z-$axis of initial (and fixed) coordinate frame
2. **Relative-axis rotation** - rotation performed about $x-$, $y-$, or $z-$axis of current (and relative) coordinate frame
   - sometimes referred to as Euler rotations
When one wants to rotate a coordinate frame about an axis, that axis can be in a fixed-frame or relative-frame.

1. Fixed-axis rotation - rotation performed about $x-$, $y-$, or $z-$axis of initial (and fixed) coordinate frame

2. Relative-axis rotation - rotation performed about $x-$, $y-$, or $z-$axis of current (and relative) coordinate frame
   - sometimes referred to as Euler rotations

Resulting orientation is quite different!
Example sequence of three consecutive rotations to compare fixed versus relative.

- **Step 1:** Rotate about the $z$-axis by $\psi$
- **Step 2:** Rotate about the $y$-axis by $\theta$
- **Step 3:** Rotate about the $x$-axis by $\phi$
Example Sequence of Rotations

Relative-axis Rotation

\[ x_1 \rightarrow z_1 \rightarrow y_2 \rightarrow z_2 \rightarrow x_3 \rightarrow y_4 \rightarrow z_3 \rightarrow x_4 \]

Fixed-axis Rotation

\[ x_1 \rightarrow z_1 \rightarrow y_1 \rightarrow x_2 \rightarrow y_2 \rightarrow z_2 \rightarrow y_3 \rightarrow z_4 \rightarrow x_4 \]
Example Sequence of Rotations

Relative-axis Rotation
Rotate about $z_1$

Fixed-axis Rotation
Rotate about $z_1$
Example Sequence of Rotations

**Relative-axis Rotation**
Rotate about $y_2$

**Fixed-axis Rotation**
Rotate about $y_1$
Example Sequence of Rotations

**Relative-axis Rotation**
Rotate about $x_3$

**Fixed-axis Rotation**
Rotate about $x_1$
Construct rotation matrix that represents composition of relative-axis rotations.

- Start with last rotation $C_4^3 = [x_4^3, y_4^3, z_4^3] = R_{x,\phi}$, and recall columns are vectors.
- To re-coordinatize vectors $x_4^3, y_4^3, z_4^3$ in frame 2, multiply each by $C_2^3 = R_{y,\theta}$. 

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In matrix form, $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = [x_4^2, y_4^2, z_4^2] = C_4^2$
Composition of Relative-axis Rotations

Construct rotation matrix that represents composition of relative-axis rotations.

- Start with last rotation $C_4^3 = [x_4^3, y_4^3, z_4^3] = R_{x, \phi}$, and recall columns are vectors.

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  In matrix form, $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = [x_4^2, y_4^2, z_4^2] = C_4^2$

  where it is noted that

  $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = C_3^2 [x_4^3, y_4^3, z_4^3] = C_3^2 C_4^3 = C_4^2$
Composition of Relative-axis Rotations

- To re-coordinatize vectors $x_4^2, y_4^2, z_4^2$ in frame 1, multiply each by $C_2^1 = R_{z,\psi}$

$$\Rightarrow [C_2^1 x_4^2, C_2^1 y_4^2, C_2^1 z_4^2] = C_2^1 [x_4^2, y_4^2, z_4^2] = C_2^1 C_4^1 = C_2^1 C_3^2 C_4^3 = C_4^1$$
To re-coordinatize vectors $x_4^2, y_4^2, z_4^2$ in frame 1, multiply each by

$C_2^1 = R_{z,\psi}$

$\Rightarrow [C_2^1 x_4^2, C_2^1 y_4^2, C_2^1 z_4^2] = C_2^1 [x_4^2, y_4^2, z_4^2] = C_2^1 C_4^2 = C_2^1 C_3^2 C_4^3 = C_4^1$

Combined sequence of relative-rotations yields

$C_4^1 = C_2^1 C_3^2 C_4^3 = R_{z,\psi} R_{y,\theta} R_{x,\phi}$

\(1st\) \(2nd\) \(3rd\)
Composition of Relative-axis Rotations

- To re-coordinatize vectors $x_2^2, y_2^2, z_2^2$ in frame 1, multiply each by $C_2^1 = R_{z,\psi}$

  $\Rightarrow [C_2^1 x_4^2, C_2^1 y_4^2, C_2^1 z_4^2] = C_2^1 [x_4^2, y_4^2, z_4^2] = C_2^1 C_4^2 = C_2^1 C_3^2 C_4^3 = C_4^1$

- Combined sequence of relative-rotations yields

  $C_4^1 = C_2^1 C_3^2 C_4^3 = R_{z,\psi} R_{y,\theta} R_{x,\phi}$

  $1st$ $2nd$ $3rd$

- Note order is left to right!

- Additional relative-rotations represented by right (post) matrix multiplies.
Rotation Matrix from Relative ZYX

For the relative-axis rotations $Z(\psi), Y(\theta), X(\phi)$

$$C_4^1 = C_2^1 C_3^2 C_4^3 = R_{z,\psi} R_{y,\theta} R_{x,\phi}$$

$$= \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$

where the notation $c_\beta = \cos(\beta)$ and $s_\beta = \sin(\beta)$ are introduced
• Development of equivalent rotation matrix for sequence of fixed-axis rotations will make use of rotation matrix’s ability to rotate a vector.

• A vector $\vec{p}$ can be rotated into a new vector via $R\vec{p}$, both in the same coordinate frame.

• We’ll once again consider the sequence $Z(\psi), Y(\theta), X(\phi)$ aka Yaw-Pitch-Roll, but this time about fixed-axes.
Composition of Fixed-axis Rotations

- First $z$–axis rotation rotates frame $\{1\}$’s basis vectors to become frame $\{2\}$’s basis vectors $[\vec{x}_1^2, \vec{y}_1^2, \vec{z}_1^2] = R_z,\psi[\vec{x}_1^1, \vec{y}_1^1, \vec{z}_1^1] = R_z,\psi$

- Second $y$–axis rotation rotates frame $\{2\}$’s basis vectors to become frame $\{3\}$’s basis vectors $[\vec{x}_1^3, \vec{y}_1^3, \vec{z}_1^3] = R_y,\theta[\vec{x}_1^2, \vec{y}_1^2, \vec{z}_1^2] = R_y,\theta R_z,\psi$

- Third $x$–axis rotation rotates frame $\{3\}$’s basis vectors to become frame $\{4\}$’s basis vector $[\vec{x}_1^4, \vec{y}_1^4, \vec{z}_1^4] = R_x,\phi[\vec{x}_1^3, \vec{y}_1^3, \vec{z}_1^3] = R_x,\phi R_y,\theta R_z,\psi$

$\Rightarrow$ $C_{14} = R_x,\phi R_y,\theta R_z,\psi$

Note order is right to left!

Additional fixed-rotations represented by left (pre) matrix multiplies.
Composition of Fixed-axis Rotations

First $z$–axis rotation rotates frame \{1\}'s basis vectors to become frame \{2\}'s basis vectors
\[
[\vec{x}_1, \vec{y}_1, \vec{z}_1] = R_{z, \psi}[\vec{x}_1, \vec{y}_1, \vec{z}_1] = R_{z, \psi}
\]

Second $y$–axis rotation rotates frame \{2\}'s basis vectors to become frame \{3\}'s basis vectors
\[
[\vec{x}_1, \vec{y}_1, \vec{z}_1] = R_{y, \theta}[\vec{x}_1, \vec{y}_1, \vec{z}_1] = R_{y, \theta}R_{z, \psi}
\]
Composition of Fixed-axis Rotations

- First \( z \)-axis rotation rotates frame \( \{1\} \)'s basis vectors to become frame \( \{2\} \)'s basis vectors
  \[
  [\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = R_{z,\psi} [\vec{x}_1^1, \vec{y}_1^1, \vec{z}_1^1] = R_{z,\psi}
  \]
- Second \( y \)-axis rotation rotates frame \( \{2\} \)'s basis vectors to become frame \( \{3\} \)'s basis vectors
  \[
  [\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = R_{y,\theta} [\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = R_{y,\theta} R_{z,\psi}
  \]
- Third \( x \)-axis rotation rotates frame \( \{3\} \)'s basis vectors to become frame \( \{4\} \)'s basis vector
  \[
  [\vec{x}_4^1, \vec{y}_4^1, \vec{z}_4^1] = R_{x,\phi} [\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = R_{x,\phi} R_{y,\theta} R_{z,\psi}
  \]
Composition of Fixed-axis Rotations

- First $z$–axis rotation rotates frame $\{1\}$’s basis vectors to become frame $\{2\}$’s basis vectors $[\vec{x}^1_2, \vec{y}^1_2, \vec{z}^1_2] = R_{z, \psi} [\vec{x}^1_1, \vec{y}^1_1, \vec{z}^1_1] = R_{z, \psi}$
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$\Rightarrow C^1_4 = R_{x, \phi} R_{y, \theta} R_{z, \psi}$

Note order is right to left!

Additional fixed-rotations represented by left (pre) matrix multiplies.
Composition of Fixed-axis Rotations

- First $z$–axis rotation rotates frame $\{1\}$’s basis vectors to become frame $\{2\}$’s basis vectors $[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = R_{z, \psi} [\vec{x}_1^1, \vec{y}_1^1, \vec{z}_1^1] = R_{z, \psi}$

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- Third $x$–axis rotation rotates frame $\{3\}$’s basis vectors to become frame $\{4\}$’s basis vector $[\vec{x}_4^1, \vec{y}_4^1, \vec{z}_4^1] = R_{x, \phi} [\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = R_{x, \phi} R_{y, \theta} R_{z, \psi}$

$$\Rightarrow C_4^1 = R_{x, \phi} R_{y, \theta} R_{z, \psi}$$

- Note order is right to left!

- Additional fixed-rotations represented by left (pre) matrix multiplies.
Composition of Fixed-axis Rotations

For the fixed-axis rotations $Z(\psi)$, $Y(\theta)$, $X(\phi)$

$$C_4^1 = R_{x,\phi} R_{y,\theta} R_{z,\psi}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta c\psi & -c\theta s\psi & s\theta \\ c\psi s\theta s\phi + c\phi s\psi & c\phi c\psi - s\theta s\phi s\psi & -c\theta s\phi \\ s\phi s\psi - c\phi c\psi s\theta & c\psi s\phi + c\phi s\theta s\psi & c\theta c\phi \end{bmatrix}$$
Composition of Fixed-axis Rotations

For the fixed-axis rotations $Z(\psi), Y(\theta), X(\phi)$

$$C^1_4 = R_{x,\phi} R_{y,\theta} R_{z,\psi}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\psi s_\theta s_\phi + c_\phi s_\psi & c_\phi c_\psi - s_\theta s_\phi s_\psi & -c_\theta s_\phi \\ s_\phi s_\psi - c_\phi c_\psi s_\theta & c_\psi s_\phi + c_\phi s_\theta s_\psi & c_\theta c_\phi \end{bmatrix}$$

which is quite different than the result for the same sequence of relative-axis rotations
Example

Find the rotation matrix representing the orientation of the coordinate frame that results from the following sequence of rotations.
Example

Find the rotation matrix representing the orientation of the coordinate frame that results from the following sequence of rotations.

1. Rotate about fixed $x$–axis by $\phi$
Example

Find the rotation matrix representing the orientation of the coordinate frame that results from the following sequence of rotations.

1. Rotate about fixed $x$–axis by $\phi$
2. Rotate about fixed $z$–axis by $\theta$
Example

Find the rotation matrix representing the orientation of the coordinate frame that results from the following sequence of rotations.

1. Rotate about fixed x–axis by $\phi$
2. Rotate about fixed z–axis by $\theta$
3. Rotate about current x–axis by $\psi$
Find the rotation matrix representing the orientation of the coordinate frame that results from the following sequence of rotations.

1. Rotate about fixed $x$–axis by $\phi$
2. Rotate about fixed $z$–axis by $\theta$
3. Rotate about current $x$–axis by $\psi$
4. Rotate about current $z$–axis by $\alpha$
Find the rotation matrix representing the orientation of the coordinate frame that results from the following sequence of rotations.

1. Rotate about fixed $x$–axis by $\phi$
2. Rotate about fixed $z$–axis by $\theta$
3. Rotate about current $x$–axis by $\psi$
4. Rotate about current $z$–axis by $\alpha$
5. Rotate about fixed $y$–axis by $\beta$
Find the rotation matrix representing the orientation of the coordinate frame that results from the following sequence of rotations.

1. Rotate about fixed \( x \)-axis by \( \phi \)
2. Rotate about fixed \( z \)-axis by \( \theta \)
3. Rotate about current \( x \)-axis by \( \psi \)
4. Rotate about current \( z \)-axis by \( \alpha \)
5. Rotate about fixed \( y \)-axis by \( \beta \)
6. Rotate about current \( y \)-axis by \( \gamma \)
Fixed vs Relative Rotations

- **Fixed-axis Rotations**
  - Multiply on the **LEFT**
  - $C_{\text{final}} = R_n \ldots R_2 R_1$

### Fixed-axis Rotation

$$C_{\text{resultant}} = R_{\text{fixed}} C_{\text{original}}$$
Fixed vs Relative Rotations

- **Fixed-axis Rotations**
  - Multiply on the **LEFT**
  - \( C_{\text{final}} = R_n \ldots R_2 R_1 \)

**Fixed-axis Rotation**

\[ C_{\text{resultant}} = R_{\text{fixed}} C_{\text{original}} \]

- **Relative-axis (Euler) Rotations**
  - Multiply on the **RIGHT**
  - \( C_{\text{final}} = R_1 R_2 \ldots R_n \)

**Relative-axis Rotation**

\[ C_{\text{resultant}} = C_{\text{original}} R_{\text{relative}} \]
Fixed vs Relative Rotations

- **Fixed-axis Rotations**
  - Multiply on the **LEFT**
  - \( C_{\text{final}} = R_n \cdots R_2 R_1 \)

### Fixed-axis Rotation

\[ C_{\text{resultant}} = R_{\text{fixed}} C_{\text{original}} \]

- **Relative-axis (Euler) Rotations**
  - Multiply on the **RIGHT**
  - \( C_{\text{final}} = R_1 R_2 \cdots R_n \)

### Relative-axis Rotation

\[ C_{\text{resultant}} = C_{\text{original}} R_{\text{relative}} \]

Two types of rotations can be composed noting order of multiplication

Kevin Wedeward, Aly El-Osery (NMT)
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The End