EE 570: Location and Navigation

Navigation Mathematics: Rotation Matrices, Part II

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Lecture Topics

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2. Examples of 3−parameter descriptions:
   - fixed-axis rotations (e.g., Roll-Pitch-Yaw/ZYX)
   - relative-axis (Euler) rotations (e.g., ZYZ, ZYX)
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   - angle and axis
3. Quaternions use 4 parameters

When one wants to rotate a coordinate frame about an axis, that axis can be in a fixed-frame or relative-frame.

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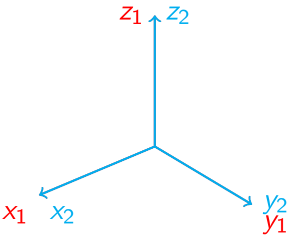
Resulting orientation is quite different!

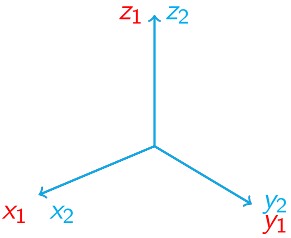
Example sequence of three consecutive rotations to compare fixed versus relative.

- **Step 1:** Rotate about the z-axis by ψ
- **Step 2:** Rotate about the y-axis by θ
- **Step 3:** Rotate about the x-axis by ϕ

Relative-axis Rotation

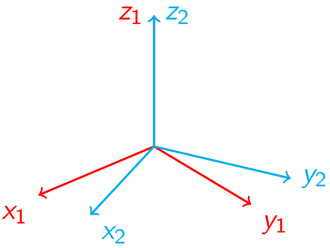


Fixed-axis Rotation



Relative-axis Rotation

Rotate about z_1



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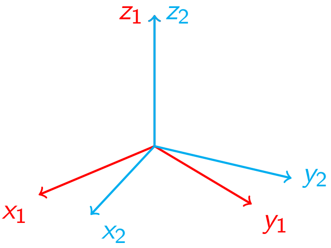
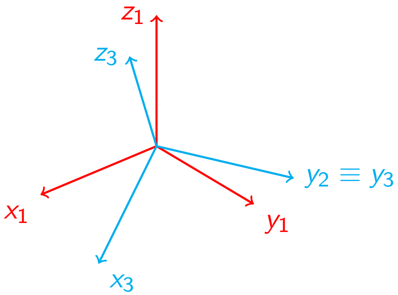


Relative-axis Rotation

Rotate about y_2



Fixed-axis Rotation

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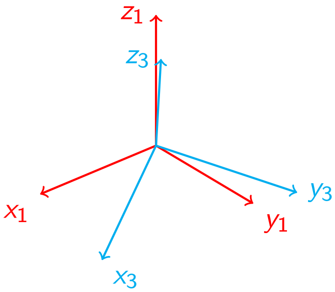
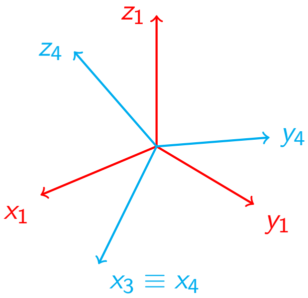


Relative-axis Rotation

Rotate about x_3



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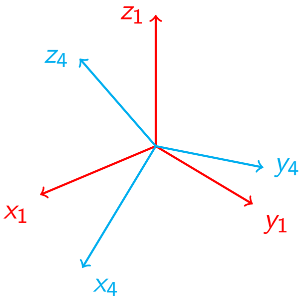

Construct rotation matrix that represents composition of relative-axis rotations.

- Start with last rotation $C_4^3 = [x_4^3, y_4^3, z_4^3] = R_{x,\phi}$, and recall columns are vectors
- To re-coordinatize vectors x_4^3, y_4^3, z_4^3 in frame 2, multiply each by $C_3^2 = R_{y,\theta}$.

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  In matrix form, $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = [x_4^2, y_4^2, z_4^2] = C_4^2$

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  where it is noted that
  $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = C_3^2 [x_4^3, y_4^3, z_4^3] = C_3^2 C_4^3 = C_4^2$

- To re-coordinatize vectors x_4^2, y_4^2, z_4^2 in frame 1, multiply each by $C_2^1 = R_{z,\psi}$

$$\Rightarrow [C_2^1 x_4^2, C_2^1 y_4^2, C_2^1 z_4^2] = C_2^1 [x_4^2, y_4^2, z_4^2] = C_2^1 C_4^2 = C_2^1 C_3^2 C_4^3 = C_4^1$$

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- Combined sequence of relative-rotations yields

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- Note order is left to right!

- Additional relative-rotations represented by right (post) matrix multiplies.

For the relative-axis rotations Z(ψ), Y(θ), X(ϕ)

$$\begin{aligned}
C_4^1 &= C_2^1 C_3^2 C_4^3 \\
&= R_{z,\psi} R_{y,\theta} R_{x,\phi} \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}
\end{aligned}$$

where the notation $c_\beta = \cos(\beta)$ and $s_\beta = \sin(\beta)$ are introduced

- Development of equivalent rotation matrix for sequence of fixed-axis rotations will make use of rotation matrix's ability to rotate a vector.
- A vector $\vec{p}$ can be rotated into a new vector via $R\vec{p}$, both in the same coordinate frame.
- We'll once again consider the sequence Z(ψ), Y(θ), X(ϕ) aka Yaw-Pitch-Roll, but this time about fixed-axes.

- First $z-$axis rotation rotates frame $\{1\}$'s basis vectors to become frame $\{2\}$'s basis vectors $[\vec{x}\,^1_2, \vec{y}\,^1_2, \vec{z}\,^1_2] = R_{z,\psi}[\vec{x}\,^1_1, \vec{y}\,^1_1, \vec{z}\,^1_1] = R_{z,\psi}$

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- Second $y-$axis rotation rotates frame $\{2\}$'s basis vectors to become frame $\{3\}$'s basis vectors $[\vec{x}^1_3, \vec{y}^1_3, \vec{z}^1_3] = R_{y,\theta}[\vec{x}^1_2, \vec{y}^1_2, \vec{z}^1_2] = R_{y,\theta}R_{z,\psi}$

Composition of Fixed-axis Rotations

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- Third $x-$axis rotation rotates frame $\{3\}$'s basis vectors to become frame $\{4\}$'s basis vector $[\vec{x}^{\,1}_{4}, \vec{y}^{\,1}_{4}, \vec{z}^{\,1}_{4}] = R_{x,\phi}[\vec{x}^{\,1}_{3}, \vec{y}^{\,1}_{3}, \vec{z}^{\,1}_{3}] = R_{x,\phi}R_{y,\theta}R_{z,\psi}$

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  $$\Rightarrow C^1_4 = \underbrace{R_{x,\phi}}_{3rd}\underbrace{R_{y,\theta}}_{2nd}\underbrace{R_{z,\psi}}_{1st}$$

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  $\Rightarrow C^1_4 = \underbrace{R_{x,\phi}}_{3rd}\underbrace{R_{y,\theta}}_{2nd}\underbrace{R_{z,\psi}}_{1st}$
- Note order is right to left!
- Additional fixed-rotations represented by left (pre) matrix multiplies.

For the fixed-axis rotations Z(ψ), Y(θ), X(ϕ)

$$
\begin{aligned}
C_4^1 &= R_{x,\phi} R_{y,\theta} R_{z,\psi} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\psi s_\theta s_\phi + c_\phi s_\psi & c_\phi c_\psi - s_\theta s_\phi s_\psi & -c_\theta s_\phi \\ s_\phi s_\psi - c_\phi c_\psi s_\theta & c_\psi s_\phi + c_\phi s_\theta s_\psi & c_\theta c_\phi \end{bmatrix}
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$$= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\psi s_\theta s_\phi + c_\phi s_\psi & c_\phi c_\psi - s_\theta s_\phi s_\psi & -c_\theta s_\phi \\ s_\phi s_\psi - c_\phi c_\psi s_\theta & c_\psi s_\phi + c_\phi s_\theta s_\psi & c_\theta c_\phi \end{bmatrix}$$

which is quite different than the result for the same sequence of relative-axis rotations

Find the rotation matrix representing the orientation of the coordinate frame that results from the following sequence of rotations.

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1. Rotate about fixed $x-$axis by ϕ
2. Rotate about fixed $z-$axis by θ

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2. Rotate about fixed $z-$axis by θ
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4. Rotate about current $z-$axis by α
5. Rotate about fixed $y-$axis by β
6. Rotate about current $y-$axis by γ

- Fixed-axis Rotations
  - Multiply on the LEFT
  - $C_{final} = R_n \dots R_2 R_1$

Fixed-axis Rotation

$$C_{resultant} = R_{fixed} C_{original}$$

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- Relative-axis (Euler) Rotations
  - Multiply on the RIGHT
  - $C_{final} = R_1 R_2 \ldots R_n$

Relative-axis Rotation

$C_{resultant} = C_{original} R_{relative}$

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  - Multiply on the LEFT
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Fixed-axis Rotation

$$C_{resultant} = R_{fixed} C_{original}$$

- Relative-axis (Euler) Rotations
  - Multiply on the RIGHT
  - $C_{final} = R_1 R_2 \ldots R_n$

Relative-axis Rotation

$$C_{resultant} = C_{original} R_{relative}$$

Two types of rotations can be composed noting order of multiplication

The End