EE 570: Location and Navigation
Navigation Equations: ECEF Mechanization

Aly El-Osery   Kevin Wedeward

Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA

In Collaboration with
Stephen Bruder
Electrical and Computer Engineering Department
Embry–Riddle Aeronautical Univesity
Prescott, Arizona, USA

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- Determine the position, velocity and attitude of the **body** frame *wrt* the **Earth** frame.
  - **Position** — Vector from the origin of the earth frame to the origin of the body frame resolved in the earth frame: $\vec{r}^{\,e}_{eb}$
  - **Velocity** — Velocity of the body frame *wrt* the earth frame resolved in the earth frame: $\vec{v}^{\,e}_{eb}$
  - **Attitude** — Orientation of the body frame *wrt* the earth frame: C^e_b

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\vec{\omega}_{ib}^{\,e} =$$

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$$\vec{\omega}\,^e_{ib} = \vec{\omega}\,^e_{ie} + \vec{\omega}\,^e_{eb}$$
$$\vec{\omega}\,^e_{eb} =$$

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$$\vec{\omega}_{ib}^{\,e} = \vec{\omega}_{ie}^{\,e} + \vec{\omega}_{eb}^{\,e}$$
$$\vec{\omega}_{eb}^{\,e} = C_b^e \vec{\omega}_{ib}^{\,b} - \vec{\omega}_{ie}^{\,e}$$

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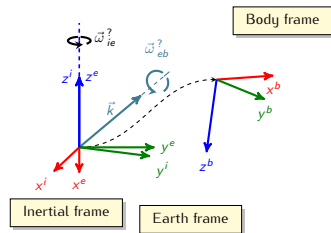$$\Omega_{eb}^e = C_b^e \Omega_{ib}^b C_e^b - \Omega_{ie}^e$$

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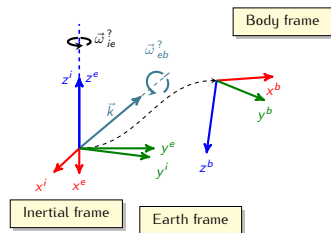
$$C_b^e(+) - C_b^e(-) \approx$$

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  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

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$$\Omega_{eb}^e = C_b^e \Omega_{ib}^b C_e^b - \Omega_{ie}^e$$

$$C_b^e(+) - C_b^e(-) \approx \Delta t \Omega_{eb}^e C_b^e(-)$$

Body frame

Inertial frame

Earth frame

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
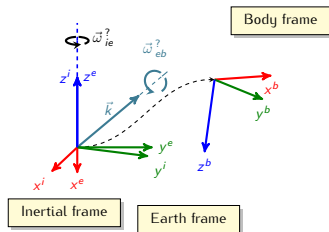- Start with angular velocity

$$\vec{\omega}_{ib}^{\,e} = \vec{\omega}_{ie}^{\,e} + \vec{\omega}_{eb}^{\,e}$$

$$\vec{\omega}_{eb}^{\,e} = C_b^e \vec{\omega}_{ib}^{\,b} - \vec{\omega}_{ie}^{\,e}$$

$$\Omega_{eb}^e = C_b^e \Omega_{ib}^b C_e^b - \Omega_{ie}^e$$

$$C_b^e(+) - C_b^e(-) \approx \Delta t \Omega_{eb}^e C_b^e(-)$$

$$C_b^e(+) \approx C_b^e(-) + \Delta t \left(C_b^e \Omega_{ib}^b C_e^b - \Omega_{ie}^e \right) C_b^e(-)$$

$$= C_b^e(-) \left(\mathcal{I} + \Omega_{ib}^b \Delta t \right) - \Omega_{ie}^e C_b^e(-) \Delta t$$

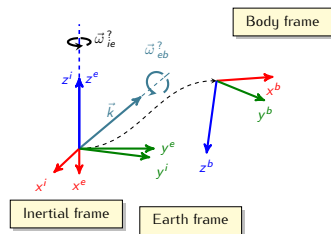Body frame

Inertial frame

Earth frame

- Body orientation frame at time "k" *wrt* time "$k - 1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\Omega_{eb}^e = C_b^e \Omega_{ib}^b C_e^b - \Omega_{ie}^e$$

$$C_b^e(+) = C_b^e(-)e^{\Omega_{eb}^b \Delta t} = e^{\Omega_{eb}^e \Delta t} C_b^e(-)$$

$$C_b^e(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\mathfrak{K}^2\right] C_b^e(-)$$

$$e^{\Omega_{eb}^e \Delta t} = e^{\mathfrak{K}\theta}$$

Body frame

Inertial frame   Earth frame

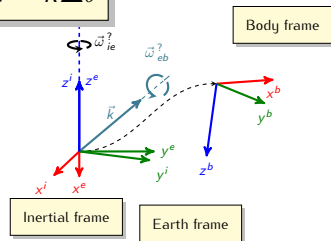- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$

$$\boxed{\vec{\omega}\,^e_{eb}\Delta t = \vec{k}\Delta\theta}$$

$$\bar{q}\,^e_b(+) = \Delta\bar{q}\,^e_b \otimes \bar{q}\,^e_b(-)$$

$$\Delta\bar{q}\,^e_b = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$

Body frame

Inertial frame

Earth frame

Need to periodically renormalize $\bar{q}$

- High fidelity

$$\vec{\omega}^{\,e}_{eb}\Delta t = \vec{k}\Delta\theta \qquad \mathfrak{K} = [\vec{k}\times]$$

$$C^e_b(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\,\mathfrak{K}^2\right] C^e_b(-) \qquad (1)$$

or

$$\bar{q}^{\,e}_{b}(+) = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}^{\,e}_{b}(-) \qquad (2)$$
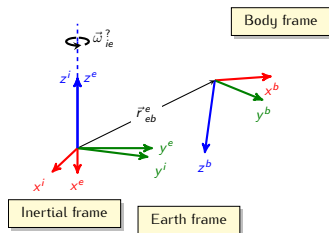
- Low fidelity

$$C^e_b(+) \approx C^e_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) - \Omega^e_{ie}C^e_b(-)\Delta t \qquad (3)$$

② Specific force transformation

- Simply coordinatize the specific force

$$\vec{f}^{\,e}_{ib} = C^e_b(+)\vec{f}^{\,b}_{ib} \tag{4}$$

③ Velocity update

$$\vec{r}^{\,i}_{ib} =$$

2. Specific force transformation
   - Simply coordinatize the specific force

$$\vec{f}^e_{ib} = C^e_b(+)\vec{f}^b_{ib} \tag{4}$$

3. Velocity update

$$\vec{r}^i_{ib} = \overset{0}{\cancel{\vec{r}^i_{ie}}} + C^i_e\vec{r}^e_{eb} \Rightarrow \vec{r}^e_{eb} = C^e_i\vec{r}^i_{ib}$$

Body frame

Inertial frame

Earth frame

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$$\vec{v}^e_{eb} =$$
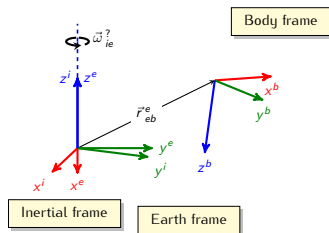

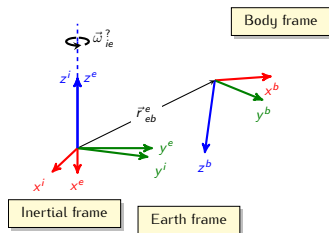Body frame

Inertial frame

Earth frame

- ② Specific force transformation
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$$\vec{f}^e_{ib} = C^e_b(+)\vec{f}^b_{ib} \qquad (4)$$

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$$\begin{aligned}
\vec{v}^e_{eb} &= \dot{\vec{r}}^e_{eb}\\
&= \dot{C}^e_i\vec{r}^i_{ib} + C^e_i\dot{\vec{r}}^i_{ib}\\
&= \Omega^e_{ei}C^e_i\vec{r}^i_{ib} + C^e_i\vec{v}^i_{ib}\\
&= -\Omega^e_{ie}\vec{r}^e_{eb} + C^e_i\vec{v}^i_{ib}
\end{aligned}$$

Body frame

Inertial frame   Earth frame

$$\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} = \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i\vec{v}^{\,i}_{ib}\right)$$

$$= -\Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{C}^e_i\vec{v}^{\,i}_{ib} + C^e_i\dot{\vec{v}}^{\,i}_{ib}$$

$$=$$

$$\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} = \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i\vec{v}^{\,i}_{ib}\right)$$

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$$= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \Omega^e_{ei}\boxed{C^e_i\vec{v}^{\,i}_{ib}} + C^e_i\vec{a}^{\,i}_{ib}$$

$$\vec{a}_{eb}^{\,e} = \dot{\vec{v}}_{eb}^{\,e} = \frac{d}{dt}\left(-\Omega_{ie}^e \vec{r}_{eb}^{\,e} + C_i^e \vec{v}_{ib}^{\,i}\right)$$

$$= -\Omega_{ie}^e \dot{\vec{r}}_{eb}^{\,e} + \dot{C}_i^e \vec{v}_{ib}^{\,i} + C_i^e \dot{\vec{v}}_{ib}^{\,i}$$

$$= -\Omega_{ie}^e \vec{v}_{eb}^{\,e} + \Omega_{ei}^e\; \boxed{C_i^e \vec{v}_{ib}^{\,i}} + C_i^e \vec{a}_{ib}^{\,i}$$

$$\boxed{C_i^e \vec{v}_{ib}^{\,i} = \Omega_{ie}^e \vec{r}_{eb}^{\,e} + \vec{v}_{eb}^{\,e}}$$

$$\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} = \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i\vec{v}^{\,i}_{ib}\right)$$

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$$\boxed{C^e_i\vec{v}^{\,i}_{ib} = \Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}}$$

$$= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \Omega^e_{ei}\,\boxed{C^e_i\vec{v}^{\,i}_{ib}} + C^e_i\vec{a}^{\,i}_{ib}$$

$$= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\left[\vec{v}^{\,e}_{eb} + \Omega^e_{ie}\vec{r}^{\,e}_{eb}\right] + C^e_i\vec{a}^{\,i}_{ib}$$

$$= -2\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \boxed{\vec{a}^{\,e}_{ib}}$$

$$\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} = \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i\vec{v}^{\,i}_{ib}\right)$$

$$= -\Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{C}^e_i\vec{v}^{\,i}_{ib} + C^e_i\dot{\vec{v}}^{\,i}_{ib}$$

$$= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \Omega_{ei}\;\boxed{C^e_i\vec{v}^{\,i}_{ib}}\; + C^e_i\vec{a}^{\,i}_{ib}$$

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$$= -2\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \boxed{\vec{a}^{\,e}_{ib}}$$

$$C^e_i\vec{v}^{\,i}_{ib} = \Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}$$

$$\vec{f}^{\,?}_{ib} = \vec{a}^{\,?}_{ib} - \vec{\gamma}^{\,?}_{ib}$$

$$\vec{a}^{\,e}_{ib} = \vec{f}^{\,e}_{ib} + \vec{\gamma}^{\,e}_{ib}$$

$$\vec{g}^{\,e}_b = \vec{\gamma}^{\,e}_{ib} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$$

$$\vec{a}^{\,e}_{ib} = \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_b + \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$$

$$\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} = \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i\vec{v}^{\,i}_{ib}\right)$$

$$= -\Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{C}^e_i\vec{v}^{\,i}_{ib} + C^e_i\dot{\vec{v}}^{\,i}_{ib}$$

$$= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \Omega^e_{ei}\;\boxed{C^e_i\vec{v}^{\,i}_{ib}}\; + C^e_i\vec{a}^{\,i}_{ib}$$

$$= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\left[\vec{v}^{\,e}_{eb} + \Omega^e_{ie}\vec{r}^{\,e}_{eb}\right] + C^e_i\vec{a}^{\,i}_{ib}$$

$$= -2\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \boxed{\vec{a}^{\,e}_{ib}}$$

$$= -2\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b}$$

$$C^e_i\vec{v}^{\,i}_{ib} = \Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}$$

$$\vec{f}^{\,?}_{ib} = \vec{a}^{\,?}_{ib} - \vec{\gamma}^{\,?}_{ib}$$

$$\vec{a}^{\,e}_{ib} = \vec{f}^{\,e}_{ib} + \vec{\gamma}^{\,e}_{ib}$$

$$\vec{g}^{\,e}_{b} = \vec{\gamma}^{\,e}_{ib} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$$

$$\vec{a}^{\,e}_{ib} = \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} + \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$$

$$\vec{a}_{eb}^{\,e} = \dot{\vec{v}}_{eb}^{\,e} = \frac{d}{dt}\left(-\Omega_{ie}^e \vec{r}_{eb}^{\,e} + C_i^e \vec{v}_{ib}^{\,i}\right)$$

$$= -\Omega_{ie}^e \dot{\vec{r}}_{eb}^{\,e} + \dot{C}_i^e \vec{v}_{ib}^{\,i} + C_i^e \dot{\vec{v}}_{ib}^{\,i}$$

$$= -\Omega_{ie}^e \vec{v}_{eb}^{\,e} + \Omega_{ei}^e \boxed{C_i^e \vec{v}_{ib}^{\,i}} + C_i^e \vec{a}_{ib}^{\,i}$$

$$= -\Omega_{ie}^e \vec{v}_{eb}^{\,e} - \Omega_{ie}^e \left[\vec{v}_{eb}^{\,e} + \Omega_{ie}^e \vec{r}_{eb}^{\,e}\right] + C_i^e \vec{a}_{ib}^{\,i}$$

$$= -2\Omega_{ie}^e \vec{v}_{eb}^{\,e} - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^{\,e} + \boxed{\vec{a}_{ib}^{\,e}}$$

$$= -2\Omega_{ie}^e \vec{v}_{eb}^{\,e} + \vec{f}_{ib}^{\,e} + \vec{g}_b^{\,e}$$

$$\boxed{C_i^e \vec{v}_{ib}^{\,i} = \Omega_{ie}^e \vec{r}_{eb}^{\,e} + \vec{v}_{eb}^{\,e}}$$

$$\boxed{\vec{f}_{ib}^{\,?} = \vec{a}_{ib}^{\,?} - \vec{\gamma}_{ib}^{\,?}}$$

$$\boxed{\vec{a}_{ib}^{\,e} = \vec{f}_{ib}^{\,e} + \vec{\gamma}_{ib}^{\,e}}$$

$$\boxed{\vec{g}_b^{\,e} = \vec{\gamma}_{ib}^{\,e} - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^{\,e}}$$

$$\boxed{\vec{a}_{ib}^{\,e} = \vec{f}_{ib}^{\,e} + \vec{g}_b^{\,e} + \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^{\,e}}$$

$$\vec{v}_{eb}^{\,e}(+) = \vec{v}_{eb}^{\,e}(-) + \vec{a}_{eb}^{\,e}\Delta t$$

$$= \vec{v}_{eb}^{\,e}(-) + \left[\vec{f}_{ib}^{\,e} + \vec{g}_b^{\,e} - 2\Omega_{ie}^i \vec{v}_{eb}^{\,e}(-)\right]\Delta t$$

$$(5)$$

④ Position update
- by simple numerical integration

$$\vec{r}^{\,e}_{eb}(+) = \vec{r}^{\,e}_{eb}(-) + \vec{v}^{\,e}_{eb}(-)\Delta t + \vec{a}^{\,e}_{eb}\frac{\Delta t^2}{2} \tag{6}$$

$$C_b^e(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\,\mathfrak{K}^2 \right] C_b^e(-)$$

or

$$C_b^e(+) \approx C_b^e(-) \left(\mathcal{I} + \Omega_{ib}^b \Delta t \right) - \Omega_{ie}^i C_b^e(-) \Delta t$$

or

$$\bar{q}_{\,b}^{\,e}(+) = \begin{bmatrix} \cos\left(\frac{\Delta\theta}{2}\right) \\ \vec{k}\sin\left(\frac{\Delta\theta}{2}\right) \end{bmatrix} \otimes \bar{q}_{\,b}^{\,e}(-)$$
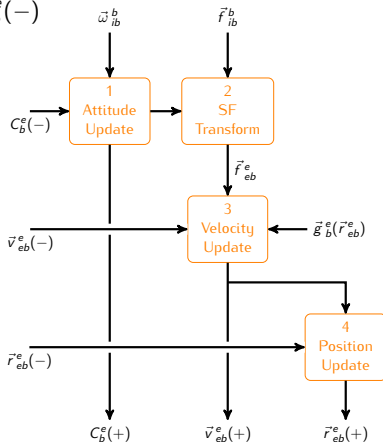
and

$$\vec{f}_{\,ib}^{\,e} = C_b^e(+)\vec{f}_{\,ib}^{\,b}$$

$$\vec{a}_{\,eb}^{\,e} = \vec{f}_{\,ib}^{\,e} + \vec{g}_{\,}^{\,e} - 2\Omega_{ie}^i \vec{v}_{\,eb}^{\,e}(-)$$

$$\vec{v}_{\,eb}^{\,e}(+) = \vec{v}_{\,eb}^{\,e}(-) + \vec{a}_{\,eb}^{\,e}\Delta t$$

$$\vec{r}_{\,eb}^{\,e}(+) = \vec{r}_{\,eb}^{\,e}(-) + \vec{v}_{\,eb}^{\,e}(-)\Delta t + \vec{a}_{\,eb}^{\,e}\frac{\Delta t^2}{2}$$

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- In continuous time notation
  - Attitude: $\dot{C}_b^e = C_b^e \Omega_{eb}^b$ or $\dot{\bar{q}}_b^e = \frac{1}{2}[\breve{\omega}_{eb}^b \circledast]\bar{q}_b^e(t)$ $\boxed{\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{eb}^b}$
  - Velocity: $\dot{\vec{v}}_{eb}^e = C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e$
  - Position: $\dot{\vec{r}}_{ib}^e = \vec{v}_{eb}^e$ $\boxed{\Omega_{eb}^b = \Omega_{ib}^b - \Omega_{ie}^b}$
- In State-space notation

$$
\begin{bmatrix} \dot{\vec{r}}_{eb}^e \\ \dot{\vec{v}}_{eb}^e \\ \dot{C}_b^i \end{bmatrix} = \begin{bmatrix} \vec{v}_{eb}^e \\ C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e \\ C_b^e \Omega_{eb}^b \end{bmatrix}
\tag{7}
$$

or

$$
\begin{bmatrix} \dot{\vec{r}}_{eb}^e \\ \dot{\vec{v}}_{eb}^e \\ \dot{\bar{q}}_b^e \end{bmatrix} = \begin{bmatrix} \vec{v}_{eb}^e \\ C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e \\ \frac{1}{2}[\breve{\omega}_{eb}^b \circledast]\bar{q}_b^e(t) \end{bmatrix}
\tag{8}
$$

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$