Lecture

Navigation Equations: Tangential Mechanization

EE 570: Location and Navigation

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Tangential Mechanization

- Determine the position, velocity and attitude of the **body** frame *wrt* the **tangential** frame.
  - **Position** — Vector from the origin of the tangential frame to the origin of the body frame resolved in the tangential frame: $\vec{r}^{\,t}_{tb}$
  - **Velocity** — Velocity of the body frame *wrt* the tangential frame resolved in the tangential frame: $\vec{v}^{\,t}_{tb}$
  - **Attitude** — Orientation of the body frame *wrt* the tangential frame: C^t_b

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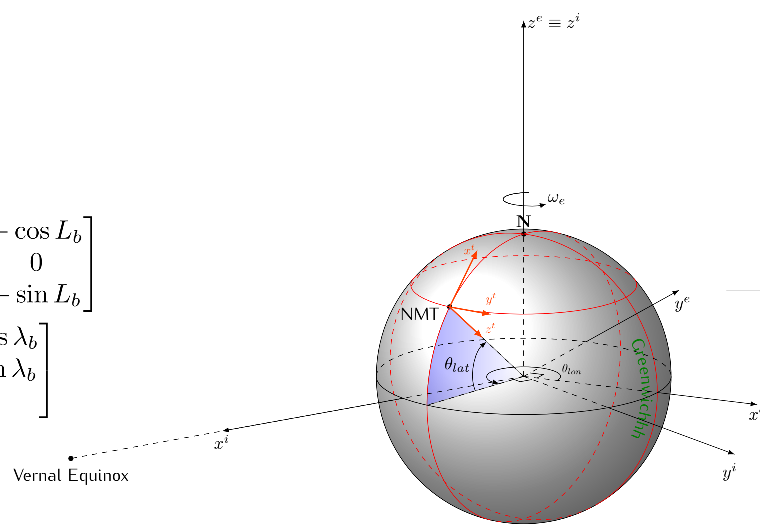
ECEF/Tangential

- Description of the tangential frame
  - Orientation of the t-frame *wrt* the e-frame

$$C^e_t = R_{(\vec{z},\lambda_b)} R_{(\vec{y},-L_b-90^\circ)}$$
$$= \begin{bmatrix} \cos\lambda_b & -\sin\lambda_b & 0 \\ \sin\lambda_b & \cos\lambda_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b & 0 & -\cos L_b \\ 0 & 1 & 0 \\ \cos L_b & 0 & -\sin L_b \end{bmatrix}$$
$$= \begin{bmatrix} -\sin L_b \cos\lambda_b & -\sin\lambda_b & -\cos L_b \cos\lambda_b \\ -\sin L_b \sin\lambda_b & \cos\lambda_b & -\cos L_b \sin\lambda_b \\ \cos L_b & 0 & -\sin L_b \end{bmatrix}$$

where geodetic Lat $= L_b$ and Geodetic Lon $= \lambda_b$



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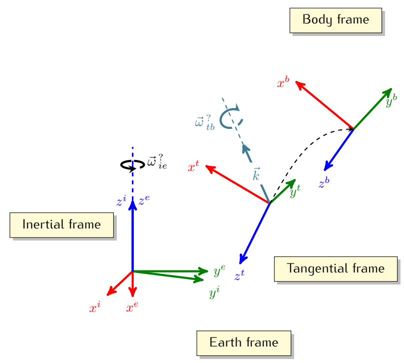
Attitude — Method A

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$

- Start with angular velocity

$$\vec{\omega}^{\,t}_{ib} = \vec{\omega}^{\,t}_{ie} + \vec{\omega}^{\,t}_{et} + \vec{\omega}^{\,t}_{tb}$$
$$\vec{\omega}^{\,t}_{tb} = C^t_b \vec{\omega}^{\,b}_{ib} - C^t_e \vec{\omega}^{\,e}_{ie}$$
$$\Omega^t_{tb} = C^t_b \Omega^b_{ib} C^b_t - C^t_e \Omega^e_{ie} C^e_t$$

$$C^t_b(+) - C^t_b(-) \approx \Delta t \Omega^t_{tb} C^t_b(-)$$
$$C^t_b(+) \approx C^t_b(-) + \Delta t \left(C^t_b \Omega^b_{ib} C^b_t - C^t_e \Omega^e_{ie} C^e_t \right) C^t_b(-)$$
$$= C^t_b(-) \left(\mathcal{I} + \Omega^b_{ib} \Delta t \right) - \Omega^t_{ie} C^t_b(-) \Delta t$$



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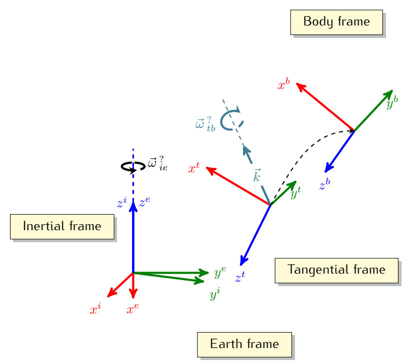
Attitude — Method B

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\Omega^t_{tb} = C^t_b \Omega^b_{ib} C^b_t - \Omega^t_{ie}$$
$$C^t_b(+) = C^t_b(-) e^{\Omega^b_{tb} \Delta t} = e^{\Omega^t_{tb} \Delta t} C^t_b(-)$$
$$C^t_b(+) = \left[\mathcal{I} + \sin(\Delta\theta) \mathfrak{K} + [1 - \cos(\Delta\theta)] \mathfrak{K}^2 \right] C^t_b(-)$$

$$e^{\Omega^t_{tb} \Delta t} = e^{\mathfrak{K}\theta}$$



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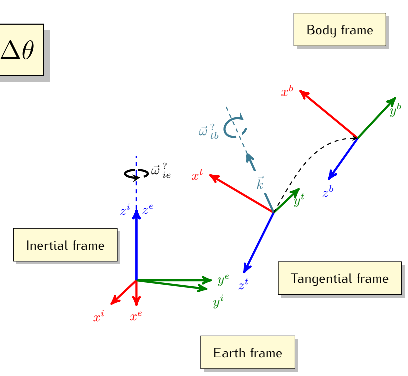
Attitude — Method C

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$

$$\vec{\omega}^{\,t}_{tb} \Delta t = \vec{k} \Delta\theta$$

$$\bar{q}^{\,t}_b(+) = \Delta\bar{q}^{\,t}_b \otimes \bar{q}^{\,t}_b(-)$$
$$\Delta\bar{q}^{\,t}_b = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$

Need to periodically renormalize $\bar{q}$



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Attitude Update— Summary

- High fidelity

$$C^t_b(+) = \left[\mathcal{I} + \sin(\Delta\theta) \mathfrak{K} + [1 - \cos(\Delta\theta)] \mathfrak{K}^2 \right] C^t_b(-) \quad (1)$$

or

$$\bar{q}^{\,t}_b(+) = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}^{\,t}_b(-) \quad (2)$$

- Low fidelity

$$C^t_b(+) \approx C^t_b(-) \left(\mathcal{I} + \Omega^b_{ib} \Delta t \right) - \Omega^t_{ie} C^t_b(-) \Delta t \quad (3)$$

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Steps 2–4

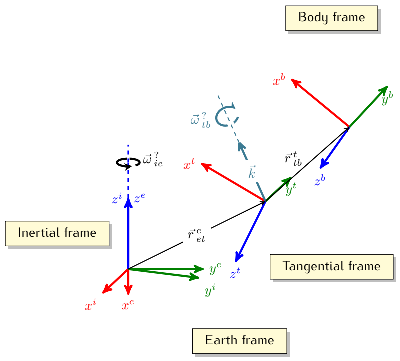
2. Specific force transformation
   - Simply coordinatize the specific force

$$\vec f^{\,t}_{ib} = C^t_b(+)\vec f^{\,b}_{ib} \tag{4}$$

3. Velocity update

$$\begin{aligned}
\vec r^{\,i}_{ib} &= \cancelto{0}{\vec r^{\,i}_{ie}} + C^i_e\vec r^{\,e}_{et} + C^i_t\vec r^{\,t}_{tb}\\
&\Rightarrow \vec r^{\,t}_{tb} = C^t_i\vec r^{\,i}_{ib} - C^t_e\vec r^{\,e}_{et}
\end{aligned}$$

$$\begin{aligned}
\vec v^{\,t}_{tb} &= \dot{\vec r}^{\,t}_{tb}\\
&= \dot C^t_i\vec r^{\,i}_{ib} + C^t_i\dot{\vec r}^{\,i}_{ib}\\
&= \Omega^t_{ei}C^t_i\vec r^{\,i}_{ib} + C^t_i\vec v^{\,i}_{ib}\\
&= -\Omega^t_{ie}(\vec r^{\,t}_{et} + \vec r^{\,t}_{tb}) + C^t_i\vec v^{\,i}_{ib}
\end{aligned}$$



Steps 2–4

$$\begin{aligned}
\vec a^{\,t}_{tb} = \dot{\vec v}^{\,t}_{tb} &= \frac{d}{dt}\left(-\Omega^t_{ie}(\vec r^{\,t}_{et} + \vec r^{\,t}_{tb}) + C^t_i\vec v^{\,i}_{ib}\right)\\
&= -\Omega^t_{ie}\dot{\vec r}^{\,t}_{tb} + \dot C^t_i\vec v^{\,i}_{ib} + C^t_i\dot{\vec v}^{\,i}_{ib}\\
&= -\Omega^t_{ie}\vec v^{\,t}_{tb} + \Omega^t_{ti}\,C^t_i\vec v^{\,i}_{ib} + C^t_i\vec a^{\,i}_{ib}\\
&= -\Omega^t_{ie}\vec v^{\,t}_{tb} - \Omega^t_{it}\left[\vec v^{\,t}_{tb} + \Omega^t_{ie}(\vec r^{\,t}_{et} + \vec r^{\,t}_{tb})\right] + C^t_i\vec a^{\,i}_{ib}\\
&= -2\Omega^t_{ie}\vec v^{\,t}_{tb} - \Omega^t_{ie}\Omega^t_{ie}\vec r^{\,t}_{eb} + \vec a^{\,t}_{ib}\\
&= -2\Omega^t_{ie}\vec v^{\,t}_{tb} + \vec f^{\,t}_{ib} + \vec g^{\,t}_{b}
\end{aligned}$$

$\dot{\vec r}^{\,t}_{et} = 0$

$\vec\omega^{\,t}_{it} = \vec\omega^{\,t}_{ie}$

$C^t_i\vec v^{\,i}_{ib} = \Omega^t_{ie}(\vec r^{\,t}_{et} + \vec r^{\,t}_{tb}) + \vec v^{\,t}_{tb}$

$\vec f^{\,?}_{ib} = \vec a^{\,?}_{ib} - \vec\gamma^{\,?}_{ib}$

$\vec a^{\,t}_{ib} = \vec f^{\,t}_{ib} + \vec\gamma^{\,t}_{ib}$

$\vec g^{\,t}_{b} = \vec\gamma^{\,t}_{ib} - \Omega^t_{ie}\Omega^t_{ie}\vec r^{\,t}_{eb}$

$\vec a^{\,t}_{ib} = \vec f^{\,t}_{ib} + \vec g^{\,t}_{b} + \Omega^t_{ie}\Omega^t_{ie}\vec r^{\,t}_{eb}$

$$\begin{aligned}
\vec v^{\,t}_{tb}(+) &= \vec v^{\,t}_{tb}(-) + \vec a^{\,t}_{tb}\Delta t\\
&= \vec v^{\,t}_{tb}(-) + \left[\vec f^{\,t}_{ib} + \vec g^{\,t}_{b} - 2\Omega^t_{ie}\vec v^{\,t}_{tb}(-)\right]\Delta t
\end{aligned} \tag{5}$$

Steps 2–4

4. Position update
   - by simple numerical integration

$$\vec r^{\,t}_{tb}(+) = \vec r^{\,t}_{tb}(-) + \vec v^{\,t}_{tb}(-)\Delta t + \vec a^{\,t}_{tb}\frac{\Delta t^2}{2} \tag{6}$$

Tangential Mechanization Summary

$$C^t_b(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right] C^t_b(-)$$

or

$$\bar{q}^{\,t}_{\,b}(+) = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}^{\,t}_{\,b}(-)$$

or

$$C^t_b(+) \approx C^t_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) - \Omega^t_{ie} C^t_b(-)\Delta t$$

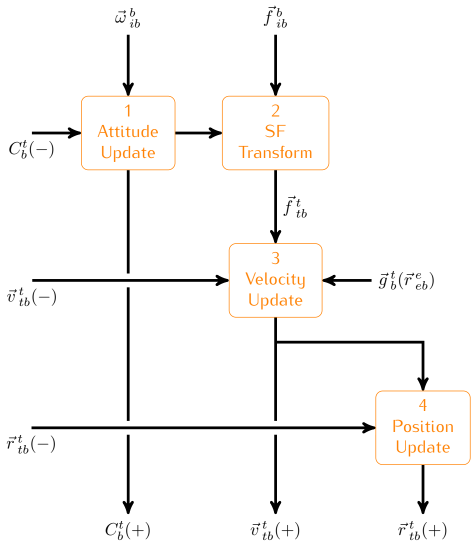
and

$$\vec{f}^{\,t}_{\,ib} = C^t_b(+)\vec{f}^{\,b}_{\,ib}$$

$$\vec{a}^{\,t}_{\,tb} = -2\Omega^t_{ie}\vec{v}^{\,t}_{\,tb} + \vec{f}^{\,t}_{\,ib} + \vec{g}^{\,t}_{\,b}$$

$$\vec{v}^{\,t}_{\,tb}(+) = \vec{v}^{\,t}_{\,tb}(-) + \left[\vec{f}^{\,t}_{\,ib} + \vec{g}^{\,t}_{\,b} - 2\Omega^t_{ie}\vec{v}^{\,t}_{\,tb}(-)\right]\Delta t$$

$$\vec{r}^{\,t}_{\,tb}(+) = \vec{r}^{\,t}_{\,tb}(-) + \vec{v}^{\,t}_{\,tb}(-)\Delta t + \vec{a}^{\,t}_{\,tb}\frac{\Delta t^2}{2}$$



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Tangential Mechanization — Continuous Case

- In continuous time notation
  - Attitude: $\dot{C}^t_b = C^t_b\Omega^b_{tb}$ or $\dot{\bar{q}}^t_b = \frac{1}{2}[\breve{\omega}^b_{tb}\circledast]\bar{q}^{\,t}_{\,b}(t)$
  - Velocity: $\dot{\vec{v}}^{\,t}_{\,tb} = C^t_b\vec{f}^{\,b}_{\,ib} + \vec{g}^{\,t}_{\,b} - 2\Omega^t_{ie}\vec{v}^{\,t}_{\,tb}$
  - Position: $\dot{\vec{r}}^{\,t}_{\,ib} = \vec{v}^{\,t}_{\,tb}$
- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}^{\,t}_{\,tb} \\ \dot{\vec{v}}^{\,t}_{\,tb} \\ \dot{C}^t_b \end{bmatrix} = \begin{bmatrix} \vec{v}^{\,t}_{\,tb} \\ C^t_b\vec{f}^{\,b}_{\,ib} + \vec{g}^{\,t}_{\,b} - 2\Omega^t_{ie}\vec{v}^{\,t}_{\,tb} \\ C^t_b\Omega^b_{tb} \end{bmatrix} \tag{7}$$

or

$$\begin{bmatrix} \dot{\vec{r}}^{\,t}_{\,tb} \\ \dot{\vec{v}}^{\,t}_{\,tb} \\ \dot{\bar{q}}^t_b \end{bmatrix} = \begin{bmatrix} \vec{v}^{\,t}_{\,tb} \\ C^t_b\vec{f}^{\,b}_{\,ib} + \vec{g}^{\,t}_{\,b} - 2\Omega^t_{ie}\vec{v}^{\,t}_{\,tb} \\ \frac{1}{2}[\breve{\omega}^b_{tb}\circledast]\bar{q}^{\,t}_{\,b}(t) \end{bmatrix} \tag{8}$$

where $\vec{\omega}^{\,b}_{\,ib} = \vec{\omega}^{\,b}_{\,ie} + \vec{\omega}^{\,b}_{\,eb}$, and $\Omega^b_{eb} = \Omega^b_{ib} - \Omega^b_{ie}$. .12

Appendix

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

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