EE 565: Position, Navigation and Timing

Error Mechanization (NAV)

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$$\dot{C}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n\right] =$$

$$\dot{C}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n\Omega_{nb}^b = [\delta\dot{\vec{\psi}}^{\,n}_{nb}\times]\hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\dot{\hat{C}}_b^n =$$

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$$\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^{\,n}\times])\hat{C}_b^n(\hat{\Omega}_{nb}^b + \delta\Omega_{ib}^b - \delta\Omega_{in}^b)$$

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$$(\mathcal{I} + [\delta\vec{\psi}_{nb}^{\,n}\times])\hat{C}_b^n\Omega_{nb}^b = [\delta\dot{\vec{\psi}}_{nb}^{\,n}\times]\hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}_{nb}^{\,n}\times])\dot{\hat{C}}_b^n =$$

$$\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^{\,n}\times])\hat{C}_b^n(\hat{\Omega}_{nb}^b + \delta\Omega_{ib}^b - \delta\Omega_{in}^b)$$

$$\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^{\,n}\times])\hat{C}_b^n\hat{\Omega}_{nb}^b + \hat{C}_b^n(\delta\Omega_{ib}^b - \delta\Omega_{in}^b) \qquad \because [\delta\vec{\psi}_{nb}^{\,n}\times]\delta\Omega_{nb}^b \approx 0$$

$$\dot{C}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n\Omega_{nb}^b = [\delta\dot{\vec{\psi}}^{\,n}_{nb}\times]\hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\dot{\hat{C}}_b^n =$$

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$=$

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$$\dot{C}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n\right] =$$

$$\begin{aligned}(\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n\Omega_{nb}^b &= [\delta\dot{\vec{\psi}}^{\,n}_{nb}\times]\hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\dot{\hat{C}}_b^n = \\ &\approx (\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n(\hat{\Omega}_{nb}^b + \delta\Omega_{ib}^b - \delta\Omega_{in}^b) \\ &\approx (\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n\hat{\Omega}_{nb}^b + \hat{C}_b^n(\delta\Omega_{ib}^b - \delta\Omega_{in}^b)\end{aligned}$$

$$[\delta\dot{\vec{\psi}}^{\,n}_{nb}\times] = \hat{C}_b^n(\delta\Omega_{nb}^b - \delta\Omega_{in}^b)\hat{C}_n^b = [\hat{C}_b^n(\delta\vec{\omega}^{\,b}_{ib} - \delta\vec{\omega}^{\,b}_{in})\times] \tag{1}$$

$$\dot{C}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n\right] =$$

$$\begin{aligned}
(\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n\Omega_{nb}^b &= [\delta\dot{\vec{\psi}}^{\,n}_{nb}\times]\hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\dot{\hat{C}}_b^n = \\
&\approx (\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n(\hat{\Omega}_{nb}^b + \delta\Omega_{ib}^b - \delta\Omega_{in}^b) \\
&\approx (\mathcal{I} + [\delta\vec{\psi}^{\,n}_{nb}\times])\hat{C}_b^n\hat{\Omega}_{nb}^b + \hat{C}_b^n(\delta\Omega_{ib}^b - \delta\Omega_{in}^b)
\end{aligned}$$

$$[\delta\dot{\vec{\psi}}^{\,n}_{nb}\times] = \hat{C}_b^n(\delta\Omega_{nb}^b - \delta\Omega_{in}^b)\hat{C}_n^b = [\hat{C}_b^n(\delta\vec{\omega}^{\,b}_{ib} - \delta\vec{\omega}^{\,b}_{in})\times] \qquad (1)$$

$$\delta\dot{\vec{\psi}}^{\,n}_{nb} = \hat{C}_b^n(\delta\omega_{ib}^b - \delta\vec{\omega}^{\,b}_{in}) \qquad (2)$$

Recall that

$$\vec{\omega}^{\,b}_{in} = C^b_n \vec{\omega}^{\,n}_{in}$$

Expressing the above equation in terms of estimates we get

$$\begin{aligned} \hat{\vec{\omega}}^b_{in} + \delta\vec{\omega}^{\,b}_{in} &= \hat{C}^b_n(\mathcal{I} - [\delta\vec{\psi}^{\,n}_{nb}\times])(\hat{\vec{\omega}}^n_{in} + \delta\vec{\omega}^{\,n}_{in}) \\ \delta\vec{\omega}^{\,b}_{in} &\approx \hat{C}^b_n(\delta\vec{\omega}^{\,n}_{in} - [\delta\vec{\psi}^{\,n}_{nb}\times]\hat{\vec{\omega}}^n_{in}) \end{aligned}$$

Substituting this result in Equation 2

$$\delta\dot{\vec{\psi}}^{\,n}_{nb} = -\hat{\vec{\Omega}}^n_{in}\delta\vec{\psi}^{\,n}_{nb} + \hat{C}^n_b\delta\vec{\omega}^{\,b}_{ib} - \delta\vec{\omega}^{\,n}_{in} \tag{3}$$

Using Taylor series and retaining the first order terms

$$\delta\vec{\omega}_{in}^{n} = \frac{\partial\hat{\vec{\omega}}_{in}^{n}}{\partial\hat{\vec{r}}_{eb}^{n}}\delta\vec{r}_{eb}^{n} + \frac{\partial\hat{\vec{\omega}}_{in}^{n}}{\partial\hat{\vec{v}}_{eb}^{n}}\delta\vec{v}_{eb}^{n}$$

where $\vec{r}_{eb}^{n} = [L_b, \lambda_b, h_b]^T$ and

$$\hat{\vec{\omega}}_{in}^{n} = \hat{\vec{\omega}}_{ie}^{n} + \hat{\vec{\omega}}_{en}^{n} = \begin{pmatrix} \omega_{ie}\cos\hat{L}_b + \frac{\hat{\vec{v}}_{eb,E}^{n}}{R_E(\hat{L}_b)+\hat{h}_b} \\ -\frac{\hat{\vec{v}}_{eb,N}^{n}}{R_N(\hat{L}_b)+\hat{h}_b} \\ -\omega_{ie}\sin\hat{L}_b - \frac{\hat{\vec{v}}_{eb,E}^{n}\tan\hat{L}_b}{R_E(\hat{L}_b)+\hat{h}_b} \end{pmatrix}$$

NAV Attitude Error (cont.)

Define terms for simplicity

$$-\hat{\vec{\Omega}}^{n}_{in} = -[(\hat{\vec{\omega}}^{n}_{ie} + \hat{\vec{\omega}}^{n}_{en})\times] = F_{\psi\psi} \tag{4}$$

$$-\frac{\partial \hat{\vec{\omega}}^{n}_{in}}{\partial \hat{\vec{v}}^{n}_{eb}} = \begin{pmatrix} 0 & -\frac{1}{R_E(\hat{L}_b)+\hat{h}_b} & 0 \\ \frac{1}{R_N(\hat{L}_b)+\hat{h}_b} & 0 & 0 \\ 0 & \frac{\tan \hat{L}_b}{R_E(\hat{L}_b)+\hat{h}_b} & 0 \end{pmatrix} = F_{\psi v} \tag{5}$$

$$-\frac{\partial \hat{\vec{\omega}}^{n}_{in}}{\partial \hat{\vec{r}}^{n}_{eb}} = \begin{pmatrix} \omega_{ie} \sin \hat{L}_b & 0 & \frac{\hat{\vec{v}}^{n}_{eb,E}}{(R_E(\hat{L}_b)+\hat{h}_b)^2} \\ 0 & 0 & -\frac{\hat{\vec{v}}^{n}_{eb,N}}{(R_N(\hat{L}_b)+\hat{h}_b)^2} \\ \omega_{ie} \cos \hat{L}_b + \frac{\hat{\vec{v}}^{n}_{eb,E}}{(R_E(\hat{L}_b)+\hat{h}_b)\cos^2 \hat{L}_b} & 0 & -\frac{\hat{\vec{v}}^{n}_{eb,E} \tan \hat{L}_b}{(R_E(\hat{L}_b)+\hat{h}_b)^2} \end{pmatrix} = F_{\psi r} \tag{6}$$

$$\delta\dot{\vec{\psi}}^{n}_{nb} = F_{\psi\psi}\delta\vec{\psi}^{n}_{nb} + F_{\psi v}\delta\vec{v}^{n}_{eb} + F_{\psi r}\delta\vec{r}^{n}_{eb} + \hat{C}^{n}_{b}\delta\vec{\omega}^{b}_{ib} \tag{7}$$

where $F_{\psi\psi}$, $F_{\psi v}$ and $F_{\psi r}$ are defined by Equations 4, 5 and 6, respectively.

$$\dot{\vec{v}}^{\,n}_{eb} = C^n_b \vec{f}^{\,b}_{ib} + \vec{g}^{\,n}_b - (\Omega^n_{en} + 2\Omega^n_{ie})\vec{v}^{\,e}_{eb} \tag{8}$$

$$\begin{aligned} \dot{\hat{\vec{v}}}^{\,n}_{eb} &= \hat{C}^n_b \hat{\vec{f}}^{\,b}_{ib} + \hat{\vec{g}}^{\,n}_b - (\hat{\Omega}^n_{en} + 2\hat{\Omega}^n_{ie})\hat{\vec{v}}^{\,n}_{eb} \\ &= (\mathcal{I} - [\delta\vec{\psi}^{\,n}_{nb}\times])C^n_b(\vec{f}^{\,b}_{ib} - \delta\vec{f}^{\,b}_{ib}) + \hat{\vec{g}}^{\,n}_b - (\hat{\Omega}^n_{en} + 2\hat{\Omega}^n_{ie})\hat{\vec{v}}^{\,n}_{eb} \end{aligned} \tag{9}$$

Computing the $\delta\dot{\vec{v}}^{\,n}_{eb}$ we obtain

$$\begin{aligned} \delta\dot{\vec{v}}^{\,n}_{eb} &= \dot{\vec{v}}^{\,n}_{eb} - \dot{\hat{\vec{v}}}^{\,n}_{eb} \\ &= [\delta\vec{\psi}^{\,n}_{nb}\times]C^n_b \vec{f}^{\,b}_{ib} + \hat{C}^n_b \delta\vec{f}^{\,b}_{ib} + \delta\vec{g}^{\,n}_b + \\ &\quad - (\hat{\Omega}^n_{en} + 2\hat{\Omega}^n_{ie})\delta\vec{v}^{\,n}_{eb} - (\delta\Omega^n_{en} + 2\delta\Omega^n_{ie})\hat{\vec{v}}^{\,n}_{eb} \end{aligned}$$

The term $-(\hat{\Omega}^n_{en} + 2\hat{\Omega}^n_{ie})$ is dervied as

$$-[(\hat{\vec{\omega}}^{\,n}_{en} + 2\hat{\vec{\omega}}^{\,n}_{ie})\times] = \begin{pmatrix} 0 & -2\omega_{ie}\sin\hat{L}_b - \frac{\hat{\vec{v}}^{\,n}_{eb,E}\tan\hat{L}_b}{R_E(\hat{L}_b)+\hat{h}_b} & \frac{\hat{\vec{v}}^{\,n}_{eb,N}}{R_N(\hat{L}_b)+\hat{h}_b} \\ 2\omega_{ie}\sin\hat{L}_b + \frac{\hat{\vec{v}}^{\,n}_{eb,E}\tan\hat{L}_b}{R_E(\hat{L}_b)+\hat{h}_b} & 0 & 2\omega_{ie}\cos\hat{L}_b + \frac{\hat{\vec{v}}^{\,n}_{eb,E}}{R_E(\hat{L}_b)+\hat{h}_b} \\ -\frac{\hat{\vec{v}}^{\,n}_{eb,N}}{R_N(\hat{L}_b)+\hat{h}_b} & -2\omega_{ie}\cos\hat{L}_b - \frac{\hat{\vec{v}}^{\,n}_{eb,E}}{R_E(\hat{L}_b)+\hat{h}_b} & 0 \end{pmatrix} = F_{\Omega\Omega} \tag{10}$$

The term $-(\delta\hat{\Omega}^n_{en} + 2\delta\hat{\Omega}^n_{ie})\hat{\vec{v}}^{\,n}_{eb}$ is dervied as

$$-(\delta\hat{\Omega}^n_{en} + 2\delta\hat{\Omega}^n_{ie})\hat{\vec{v}}^{\,n}_{eb} = \frac{\partial F_{\Omega\Omega}}{\partial \hat{\vec{r}}^{\,n}_{eb}}\delta\vec{r}^{\,n}_{eb}\hat{\vec{v}}^{\,n}_{eb} + \frac{\partial F_{\Omega\Omega}}{\partial \hat{\vec{v}}^{\,n}_{eb}}\delta\vec{v}^{\,n}_{eb}\hat{\vec{v}}^{\,n}_{eb} \tag{11}$$

$$\delta \vec{g}\,^n_b \approx \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \delta h_b \tag{12}$$

After tons of algebra

$$\delta \dot{\vec{v}}_{eb}^{n} = F_{v\psi}\delta\vec{\psi}_{nb}^{n} + F_{vv}\delta\vec{v}_{eb}^{n} + F_{vr}\delta\vec{r}_{eb}^{n} + \hat{C}_b^n\delta\vec{f}_{ib}^{b} \quad (13)$$

where $F_{v\psi}$, F_{vv} and F_{vr} are defined by Equations 14, 15 and 16, respectively.

$F_{v\psi}$, F_{vv} and F_{vr}

$$F_{v\psi} = -\left[(\hat{C}_b^n \hat{\vec{f}}_{ib}^{\,b}) \times \right] \tag{14}$$

$$F_{vv} = \begin{pmatrix} \frac{\hat{v}^n_{eb,D}}{(R_N(\hat{L}_b)+\hat{h}_b)} & -\frac{2\hat{v}^n_{eb,E}\tan\hat{L}_b}{(R_E(\hat{L}_b)+\hat{h}_b)} - 2\omega_{ie}\sin\hat{L}_b & \frac{\hat{v}^n_{eb,N}}{R_N(\hat{L}_b)+\hat{h}_b} \\ \frac{\hat{v}^n_{eb,E}\tan\hat{L}_b}{(R_E(\hat{L}_b)+\hat{h}_b)} + 2\omega_{ie}\sin\hat{L}_b & \frac{\hat{v}^n_{eb,N}\tan\hat{L}_b + \hat{v}^n_{eb,D}}{R_N(\hat{L}_b)+\hat{h}_b} & \frac{\hat{v}^n_{eb,E}}{(R_E(\hat{L}_b)+\hat{h}_b)} + 2\omega_{ie}\cos\hat{L}_b \\ -\frac{2\hat{v}^n_{eb,N}}{R_N(\hat{L}_b)+\hat{h}_b} & -\frac{2\hat{v}^n_{eb,E}}{(R_E(\hat{L}_b)+\hat{h}_b)} - 2\omega_{ie}\cos\hat{L}_b & 0 \end{pmatrix} \tag{15}$$

$$F_{vr} = \begin{pmatrix} -\frac{(\hat{v}^n_{eb,E})^2\sec^2\hat{L}_b}{R_E(\hat{L}_b)+\hat{h}_b} - 2\hat{v}^n_{eb,E}\omega_{ie}\cos\hat{L}_b & 0 & \frac{(\hat{v}^n_{eb,E})^2\tan\hat{L}_b}{(R_E(\hat{L}_b)+\hat{h}_b)^2} - \frac{\hat{v}^n_{eb,N}\hat{v}^n_{eb,D}}{(R_N(\hat{L}_b)+\hat{h}_b)^2} \\ \frac{\hat{v}^n_{eb,N}\hat{v}^n_{eb,E}\sec^2\hat{L}_b}{R_E(\hat{L}_b)+\hat{h}_b} + 2\hat{v}^n_{eb,N}\omega_{ie}\cos\hat{L}_b - 2\hat{v}^n_{eb,D}\omega_{ie}\sin\hat{L}_b & 0 & -\frac{\hat{v}^n_{eb,N}\hat{v}^n_{eb,E}\tan\hat{L}_b + \hat{v}^n_{eb,E}\hat{v}^n_{eb,D}}{(R_E(\hat{L}_b)+\hat{h}_b)^2} \\ 2\hat{v}^n_{eb,E}\omega_{ie}\sin\hat{L}_b & 0 & \left(\frac{(\hat{v}^n_{eb,E})^2}{(R_E(\hat{L}_b)+\hat{h}_b)^2} + \frac{(\hat{v}^n_{eb,N})^2}{(R_N(\hat{L}_b)+\hat{h}_b)^2} - \frac{2g_0(\hat{L}_b)}{r^e_{es}(\hat{L}_b)}\right) \end{pmatrix} \tag{16}$$

$$\dot{\hat{\vec{r}}}^{n}_{eb} = \begin{pmatrix} \frac{\hat{\vec{v}}^{n}_{eb,N}}{R_N(\hat{L}_b)+\hat{h}_b} \\ \frac{\hat{\vec{v}}^{n}_{eb,E}}{\cos\hat{L}_b(R_E(\hat{L}_b)+\hat{h}_b)}) \\ -\hat{\vec{v}}^{n}_{eb,D} \end{pmatrix} \tag{17}$$

Computing $\delta\dot{\vec{r}}^{n}_{eb}$ using taylor series expansion and retaining only the first order terms

$$\delta\dot{\vec{r}}^{n}_{eb} = F_{r\psi}\delta\vec{\psi}^{n}_{nb} + F_{rv}\delta\vec{v}^{n}_{eb} + F_{rr}\delta\vec{r}^{n}_{eb} \tag{18}$$

where $F_{r\psi}$, F_{rv} and F_{rr} are defined by Equations 19, 20 and 21, respectively.

Position Cont.

$F_{r\psi}$, F_{rv} and F_{rr}

$$F_{r\psi} = 0_{3\times 3} \tag{19}$$

$$\mathrm{F}_{rv} = \begin{pmatrix} \frac{1}{R_N(\hat{L}_b)+\hat{h}_b} & 0 & 0 \\ 0 & \frac{1}{(R_E(\hat{L}_b)+\hat{h}_b)\cos\hat{L}_b} & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{20}$$

$$\mathrm{F}_{rr} = \begin{pmatrix} 0 & 0 & -\frac{\hat{v}^n_{eb,N}}{(R_N(\hat{L}_b)+\hat{h}_b)^2} \\ \frac{\hat{v}^n_{eb,E}\tan\hat{L}_b}{(R_N(\hat{L}_b)+\hat{h}_b)\cos\hat{L}_b} & 0 & -\frac{\hat{v}^n_{eb,E}}{(R_N(\hat{L}_b)+\hat{h}_b)^2\cos\hat{L}_b} \\ 0 & 0 & 0 \end{pmatrix} \tag{21}$$

$$\begin{pmatrix} \delta\dot{\vec{\psi}}^{\,n}_{nb} \\ \delta\dot{\vec{v}}^{\,n}_{nb} \\ \delta\dot{\vec{r}}^{\,n}_{nb} \end{pmatrix} = \begin{pmatrix} F_{\psi\psi} & F_{\psi v} & F_{\psi r} \\ F_{v\psi} & F_{vv} & F_{vr} \\ 0_{3\times 3} & F_{rv} & F_{rr} \end{pmatrix} \begin{pmatrix} \delta\vec{\psi}^{n}_{nb} \\ \delta\vec{v}^{n}_{nb} \\ \delta\vec{r}^{n}_{nb} \end{pmatrix} + \begin{pmatrix} 0 & \hat{C}^{n}_{b} \\ \hat{C}^{n}_{b} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta\vec{f}^{\,b}_{ib} \\ \delta\vec{\omega}^{\,b}_{ib} \end{pmatrix} \tag{22}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

Nothing above

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

Use "tilde"

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

"Use hat"

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

Linearization using Taylor Series Expansion

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

Linearization using Taylor Series Expansion

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Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{23}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$
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Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \ f(\hat{\vec{x}}, t) \ + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T$$

$$\approx \ \dot{\hat{\vec{x}}} \ + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x}$$

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$$\approx \dot{\hat{\vec{x}}} + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$
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$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{23}$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \; & f(\hat{\vec{x}}, t) \; + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x} = \hat{\vec{x}}} \delta\vec{x} + H.O.T \\ \approx \; & \dot{\hat{\vec{x}}} \; + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x} = \hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

$$\Rightarrow \delta\dot{\vec{x}} \approx \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x} = \hat{\vec{x}}} \delta\vec{x} \tag{24}$$

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}^{\,b}_{ib}$ and $\tilde{\vec{\omega}}^{\,b}_{ib}$, respectively, will be modeled as

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta\vec{f}^{\,b}_{ib} \tag{25}$$

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{\omega}^{\,b}_{ib} + \Delta\vec{\omega}^{\,b}_{ib} \tag{26}$$

where $\vec{f}^{\,b}_{ib}$ and $\vec{\omega}^{\,b}_{ib}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}^{\,b}_{ib}$ and $\Delta\vec{\omega}^{\,b}_{ib}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

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$$\left.\begin{aligned} \tilde{\vec{f}}^{\,b}_{ib} &= \vec{f}^{\,b}_{ib} + \Delta\vec{f}^{\,b}_{ib} \qquad (25) \\ \tilde{\vec{\omega}}^{\,b}_{ib} &= \vec{\omega}^{\,b}_{ib} + \Delta\vec{\omega}^{\,b}_{ib} \qquad (26) \end{aligned}\right\} \text{these terms may be expanded further}$$

where $\vec{f}^{\,b}_{ib}$ and $\vec{\omega}^{\,b}_{ib}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}^{\,b}_{ib}$ and $\Delta\vec{\omega}^{\,b}_{ib}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

Accelerometers

$$\tilde{\vec{f}}\,^b_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}\,^b_{ib} + \vec{nl}_a + \vec{w}_a \tag{27}$$

Gyroscopes

$$\tilde{\vec{\omega}}\,^b_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}\,^b_{ib} + G_g \vec{f}\,^b_{ib} + \vec{w}_g \tag{28}$$

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{27}$$

Biases

Gyroscopes

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{\,b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{28}$$

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{27}$$

Misalignment and SF Errors

Gyroscopes

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{\,b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{28}$$

Accelerometers

$$\tilde{\vec{f}}\,^b_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}\,^b_{ib} + \vec{nl}_a + \vec{w}_a \tag{27}$$

Non-linearity

Gyroscopes

$$\tilde{\vec{\omega}}\,^b_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}\,^b_{ib} + G_g\vec{f}\,^b_{ib} + \vec{w}_g \tag{28}$$

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{27}$$

G-Sensitivity

Gyroscopes

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{\,b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{28}$$

Accelerometers

$$\tilde{\vec{f}}\,^b_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}\,^b_{ib} + \vec{nl}_a + \vec{w}_a \tag{27}$$

Noise

Gyroscopes

$$\tilde{\vec{\omega}}\,^b_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}\,^b_{ib} + G_g\vec{f}\,^b_{ib} + \vec{w}_g \tag{28}$$

- Position error

$$\delta\vec{r}^{\,\gamma}_{\beta b} = \vec{r}^{\,\gamma}_{\beta b} - \hat{\vec{r}}^{\,\gamma}_{\beta b} \tag{29}$$

- Velocity error

$$\delta\vec{v}^{\,\gamma}_{\beta b} = \vec{v}^{\,\gamma}_{\beta b} - \hat{\vec{v}}^{\,\gamma}_{\beta b} \tag{30}$$

- Specific force errors

$$\delta\vec{f}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} - \hat{\vec{f}}^{\,b}_{ib} \tag{31}$$

$$\Delta_e\vec{f}^{\,b}_{ib} = \Delta\vec{f}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = -\delta\vec{f}^{\,b}_{ib} \tag{32}$$

- Angular rate errors

$$\delta\vec{\omega}^{\,b}_{ib} = \vec{\omega}^{\,b}_{ib} - \hat{\vec{\omega}}^{\,b}_{ib} \tag{33}$$

$$\Delta_e\vec{\omega}^{\,b}_{ib} = \Delta\vec{\omega}^{\,b}_{ib} - \Delta\hat{\vec{\omega}}^{\,b}_{ib} = -\delta\vec{\omega}^{\,b}_{ib} \tag{34}$$

Define

$$\delta C_b^\gamma = C_b^\gamma \hat{C}_\gamma^b = e^{[\delta\vec{\psi}^\gamma_{\gamma b}\times]} \approx \mathcal{I} + [\delta\vec{\psi}^\gamma_{\gamma b}\times] \tag{35}$$

This is the error in attitude resulting from errors in estimating the angular rates.

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])C_b^\gamma \quad (36)$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])\hat{C}_b^\gamma \quad (37)$$

Similarly we can attempt to estimate the specific force and angular rate by applying correction based on our estimate of the error.

$$\hat{\vec{f}}_{ib}^{\,b} = \tilde{\vec{f}}_{ib}^{\,b} - \Delta\hat{\vec{f}}_{ib}^{\,b} \tag{38}$$

$$\hat{\vec{\omega}}_{ib}^{\,b} = \tilde{\vec{\omega}}_{ib}^{\,b} - \Delta\hat{\vec{\omega}}_{ib}^{\,b} \tag{39}$$

where $\hat{\vec{f}}_{ib}^{\,b}$ and $\hat{\vec{\omega}}_{ib}^{\,b}$ are the accelerometer and gyroscope estimated calibration values, respectively.