EE 565: Position, Navigation and Timing

Error Mechanization (Tangential)

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Tangential Mechanization — Continuous Case

$$\Omega^b_{tb} = \Omega^b_{ib} - \Omega^b_{ie}$$

$$\vec{\omega}^b_{ib} = \vec{\omega}^b_{ie} + \vec{\omega}^b_{tb}$$

- In continuous time notation
  - Attitude: $\dot{C}^t_b = C^t_b \Omega^b_{tb}$
  - Velocity: $\dot{\vec{v}}^{\,t}_{tb} = C^t_b \vec{f}^{\,b}_{ib} + \vec{g}^t_b - 2\Omega^t_{ie} \vec{v}^{\,t}_{tb}$
  - Position: $\dot{\vec{r}}^{\,t}_{ib} = \vec{v}^{\,t}_{tb}$
- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}^{\,t}_{tb} \\ \dot{\vec{v}}^{\,t}_{tb} \\ \dot{C}^t_b \end{bmatrix} = \begin{bmatrix} \vec{v}^t_{tb} \\ C^t_b \vec{f}^b_{ib} + \vec{g}^t_b - 2\Omega^t_{ie} \vec{v}^{\,t}_{tb} \\ C^t_b \Omega^b_{tb} \end{bmatrix} \tag{1}$$

$$\Omega_{tb}^{b} = \Omega_{ib}^{b} - \Omega_{ie}^{b}$$

$$\vec{\omega}_{ib}^{b} = \vec{\omega}_{ie}^{b} + \vec{\omega}_{tb}^{b}$$

- In continuous time notation
  - Attitude: $\dot{C}_b^t = C_b^t \Omega_{tb}^b$
  - Velocity: $\dot{\vec{v}}_{tb}^{\,t} = C_b^t \vec{f}_{ib}^{\,b} + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^{\,t}$
  - Position: $\dot{\vec{r}}_{ib}^{\,t} = \vec{v}_{tb}^{\,t}$
- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}_{tb}^{\,t} \\ \dot{\vec{v}}_{tb}^{\,t} \\ \dot{C}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}_{tb}^{t} \\ C_b^t \vec{f}_{ib}^{b} + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^{\,t} \\ C_b^t \Omega_{tb}^b \end{bmatrix} \tag{1}$$

Question

What is the effect of sensor noise in the measurements $\vec{\omega}_{ib}^{b}$ and $\vec{f}_{ib}^{b}$ on the navigation solution position, velocity and attitude?

$$\dot{C}_b^t = C_b^t \Omega_{tb}^b \; = C_b^t(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t\right] =$$

Tangential Attitude Error

$$\dot{C}_b^t = C_b^t \Omega_{tb}^b \;\; = C_b^t (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \right] =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \Omega_{tb}^b = [\delta \dot{\vec{\psi}}_{tb}^t \times] \hat{C}_b^t + (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \dot{\hat{C}}_b^t =$$

$$\dot{C}_b^t = C_b^t \Omega_{tb}^b \;= C_b^t(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t \Omega_{tb}^b = [\delta\dot{\vec{\psi}}_{tb}^t \times]\hat{C}_b^t + (\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\dot{\hat{C}}_b^t =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t(\hat{\Omega}_{tb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$\dot{C}_b^t = C_b^t \Omega_{tb}^b \ = C_b^t(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{tb}^t\times])\hat{C}_b^t\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t\times])\hat{C}_b^t\Omega_{tb}^b = [\delta\dot{\vec{\psi}}_{tb}^t\times]\hat{C}_b^t + (\mathcal{I} + [\delta\vec{\psi}_{tb}^t\times])\dot{\hat{C}}_b^t =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t\times])\hat{C}_b^t(\hat{\Omega}_{tb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t\times])\hat{C}_b^t\hat{\Omega}_{tb}^b + \hat{C}_b^t(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$\because [\delta\vec{\psi}_{tb}^t\times]\delta\Omega_{tb}^b \approx 0$$

$$\dot{C}_b^t = C_b^t \Omega_{tb}^b = C_b^t(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t \Omega_{tb}^b = [\delta\dot{\vec{\psi}}_{tb}^t \times]\hat{C}_b^t + (\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\dot{\hat{C}}_b^t =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t(\hat{\Omega}_{tb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t \hat{\Omega}_{tb}^b + \hat{C}_b^t(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$=$

$$\dot{C}_b^t = C_b^t \Omega_{tb}^b = C_b^t(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t \Omega_{tb}^b = [\delta\dot{\vec{\psi}}_{tb}^t \times]\hat{C}_b^t + (\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\dot{\hat{C}}_b^t =$$
$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t(\hat{\Omega}_{tb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$
$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t\hat{\Omega}_{tb}^b + \hat{C}_b^t(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$[\delta\dot{\vec{\psi}}_{tb}^t \times] = \hat{C}_b^t(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b)\hat{C}_t^b = [\hat{C}_b^t(\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{ie}^b)\times] \quad (2)$$

$$\dot{C}_b^t = C_b^t \Omega_{tb}^b \;= C_b^t(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t \Omega_{tb}^b = [\delta\dot{\vec{\psi}}_{tb}^t \times]\hat{C}_b^t + (\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\dot{\hat{C}}_b^t =$$
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$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t\hat{\Omega}_{tb}^b + \hat{C}_b^t(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$[\delta\dot{\vec{\psi}}_{tb}^t \times] = \hat{C}_b^t(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b)\hat{C}_t^b = [\hat{C}_b^t(\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{ie}^b)\times] \tag{2}$$

$$\delta\dot{\vec{\psi}}_{tb}^t = \hat{C}_b^t(\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{ie}^b) \tag{3}$$

Tangential Attitude Error (cont.)

$$
\begin{aligned}
\delta\dot{\vec{\psi}}^t_{tb} &= \hat{C}^t_b(\delta\vec{\omega}^b_{ib} - \delta\vec{\omega}^b_{ie}) \\
&= \hat{C}^t_b\delta\vec{\omega}^b_{ib} - \hat{C}^t_b(\vec{\omega}^b_{ie} - \hat{\vec{\omega}}^b_{ie}) \\
&= \hat{C}^t_b\delta\vec{\omega}^b_{ib} - (\hat{C}^t_b C^b_t - \mathcal{I})\vec{\omega}^t_{ie} \\
&= \hat{C}^t_b\delta\vec{\omega}^b_{ib} + \delta\vec{\psi}^t_{tb} \times \vec{\omega}^t_{ie}
\end{aligned}
$$

$$
\begin{aligned}
\delta\dot{\vec{\psi}}^t_{tb} &= \hat{C}^t_b(\delta\vec{\omega}^b_{ib} - \delta\vec{\omega}^b_{ie}) \\
&= \hat{C}^t_b\delta\vec{\omega}^b_{ib} - \hat{C}^t_b(\vec{\omega}^b_{ie} - \hat{\vec{\omega}}^b_{ie}) \\
&= \hat{C}^t_b\delta\vec{\omega}^b_{ib} - (\hat{C}^t_b C^b_t - \mathcal{I})\vec{\omega}^t_{ie} \\
&= \hat{C}^t_b\delta\vec{\omega}^b_{ib} + \delta\vec{\psi}^t_{tb} \times \vec{\omega}^t_{ie}
\end{aligned}
$$

$$
\delta\dot{\vec{\psi}}^t_{tb} = \hat{C}^t_b\delta\vec{\omega}^b_{ib} - \vec{\Omega}^t_{ie}\delta\vec{\psi}^t_{tb} \tag{4}
$$

Velocity

$$\dot{\vec{v}}^{\,t}_{tb} = C^t_b \vec{f}^{\,b}_{ib} + \vec{g}^t_b - 2\Omega^t_{ie}\vec{v}^{\,t}_{tb} \tag{5}$$

$$\begin{aligned}\dot{\hat{\vec{v}}}^{\,t}_{tb} &= \hat{C}^t_b \hat{\vec{f}}^{\,b}_{ib} + \hat{\vec{g}}^t_b - 2\Omega^t_{ie}\hat{\vec{v}}^{\,t}_{tb} \\ &= (\mathcal{I} - [\delta\vec{\psi}^t_{tb}\times])C^t_b(\vec{f}^{\,b}_{ib} - \delta\vec{f}^{\,b}_{ib}) + \hat{\vec{g}}^t_b - 2\Omega^t_{ie}\hat{\vec{v}}^{\,t}_{tb}\end{aligned} \tag{6}$$

$$\dot{\vec{v}}^{\,t}_{tb} = C^t_b \vec{f}^{\,b}_{ib} + \vec{g}^t_b - 2\Omega^t_{ie}\vec{v}^{\,t}_{tb} \tag{5}$$

$$\begin{aligned} \dot{\hat{\vec{v}}}^{\,t}_{tb} &= \hat{C}^t_b \hat{\vec{f}}^{\,b}_{ib} + \hat{\vec{g}}^t_b - 2\Omega^t_{ie}\hat{\vec{v}}^{\,t}_{tb} \\ &= (\mathcal{I} - [\delta\vec{\psi}^t_{tb}\times])C^t_b(\vec{f}^{\,b}_{ib} - \delta\vec{f}^{\,b}_{ib}) + \hat{\vec{g}}^t_b - 2\Omega^t_{ie}\hat{\vec{v}}^{\,t}_{tb} \end{aligned} \tag{6}$$

$$\begin{aligned} \delta\dot{\vec{v}}^{\,t}_{tb} = \dot{\vec{v}}^{\,t}_{tb} - \dot{\hat{\vec{v}}}^{\,t}_{tb} &= [\delta\vec{\psi}^t_{tb}\times]C^t_b\vec{f}^{\,b}_{ib} + \hat{C}^t_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{g}^t_b - 2\Omega^t_{ie}\delta\vec{v}^{\,t}_{tb} \\ &= [\delta\vec{\psi}^t_{tb}\times]\hat{C}^t_b\hat{\vec{f}}^{\,b}_{ib} + \hat{C}^t_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{g}^t_b - 2\Omega^t_{ie}\delta\vec{v}^{\,t}_{tb} \end{aligned}$$

$$\dot{\vec{v}}^{\,t}_{tb} = C^t_b \vec{f}^{\,b}_{ib} + \vec{g}^t_b - 2\Omega^t_{ie}\vec{v}^{\,t}_{tb} \tag{5}$$

$$\begin{aligned}\dot{\hat{\vec{v}}}^{\,t}_{tb} &= \hat{C}^t_b \hat{\vec{f}}^{\,b}_{ib} + \hat{\vec{g}}^t_b - 2\Omega^t_{ie}\hat{\vec{v}}^{\,t}_{tb} \\ &= (\mathcal{I} - [\delta\vec{\psi}^t_{tb}\times])C^t_b(\vec{f}^{\,b}_{ib} - \delta\vec{f}^{\,b}_{ib}) + \hat{\vec{g}}^t_b - 2\Omega^t_{ie}\hat{\vec{v}}^{\,t}_{tb}\end{aligned} \tag{6}$$

$$\begin{aligned}\delta\dot{\vec{v}}^{\,t}_{tb} = \dot{\vec{v}}^{\,t}_{tb} - \dot{\hat{\vec{v}}}^{\,t}_{tb} &= [\delta\vec{\psi}^t_{tb}\times]C^t_b\vec{f}^{\,b}_{ib} + \hat{C}^t_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{g}^t_b - 2\Omega^t_{ie}\delta\vec{v}^{\,t}_{tb} \\ &= [\delta\vec{\psi}^t_{tb}\times]\hat{C}^t_b\hat{\vec{f}}^{\,b}_{ib} + \hat{C}^t_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{g}^t_b - 2\Omega^t_{ie}\delta\vec{v}^{\,t}_{tb}\end{aligned}$$

$$\delta\dot{\vec{v}}^{\,t}_{tb} = -[\hat{C}^t_b\hat{\vec{f}}^{\,b}_{ib}\times]\delta\vec{\psi}^t_{tb} + \hat{C}^t_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{g}^t_b - 2\Omega^t_{ie}\hat{\vec{v}}^{\,t}_{tb} \tag{7}$$

$$\delta\vec{g}^t_b \approx \hat{C}^t_e \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^{\,e}_{eb}}{|\hat{\vec{r}}^{\,e}_{eb}|^2} (\hat{\vec{r}}^{\,e}_{eb})^T \hat{C}^e_t \delta\vec{r}^{\,t}_{tb} \tag{8}$$

$$\dot{\vec{r}}^{\,t}_{tb} = \vec{v}^{\,t}_{tb} \tag{9}$$

Position

$$\dot{\vec{r}}^{\,t}_{tb} = \vec{v}^{\,t}_{tb} \tag{9}$$

$$\delta\dot{\vec{r}}^{\,t}_{tb} = \delta\vec{v}^{\,t}_{tb} \tag{10}$$

$$\begin{pmatrix} \delta\dot{\vec{\psi}}^t_{tb} \\ \delta\dot{\vec{v}}^{\,t}_{tb} \\ \delta\dot{\vec{r}}^{\,t}_{tb} \end{pmatrix} = \begin{bmatrix} -\Omega^t_{ie} & 0_{3\times3} & 0_{3\times3} \\ -[\hat{C}^t_b\hat{\vec{f}}^{\,b}_{ib}\times] & -2\Omega^t_{ie} & \hat{C}^t_e \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^{\,e}_{eb}}{|\hat{\vec{r}}^{\,e}_{eb}|^2} (\hat{\vec{r}}^{\,e}_{eb})^T \hat{C}^e_t \\ 0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{pmatrix} \delta\vec{\psi}^t_{tb} \\ \delta\vec{v}^{\,t}_{tb} \\ \delta\vec{r}^{\,t}_{tb} \end{pmatrix} + \begin{bmatrix} 0 & \hat{C}^t_b \\ \hat{C}^t_b & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \delta\vec{f}^{\,b}_{ib} \\ \delta\vec{\omega}^{\,b}_{ib} \end{pmatrix} \tag{11}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value $\vec{x}$ — Nothing above
- Measured value $\tilde{\vec{x}}$
- Estimated or computed value $\hat{\vec{x}}$
- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$ → Use "tilde"

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

"Use hat"

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

Linearization using Taylor Series Expansion

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

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Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{12}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$
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Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}}, t) + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T$$
$$\approx \dot{\hat{\vec{x}}} + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x}$$

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Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \; f(\hat{\vec{x}}, t) \; + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T$$

$$\approx \; \dot{\hat{\vec{x}}} \; + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x}$$

$$\Rightarrow \delta\dot{\vec{x}} \approx \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \tag{13}$$

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}_{ib}^{b}$ and $\tilde{\vec{\omega}}_{ib}^{b}$, respectively, will be modeled as

$$\tilde{\vec{f}}_{ib}^{b} = \vec{f}_{ib}^{\,b} + \Delta\vec{f}_{ib}^{\,b} \tag{14}$$

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{\omega}_{ib}^{\,b} + \Delta\vec{\omega}_{ib}^{\,b} \tag{15}$$

where $\vec{f}_{ib}^{\,b}$ and $\vec{\omega}_{ib}^{\,b}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}_{ib}^{\,b}$ and $\Delta\vec{\omega}_{ib}^{\,b}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

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these terms may be expanded further

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Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{\,b} + \vec{nl}_a + \vec{w}_a \tag{16}$$

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{\,b} + G_g\vec{f}_{ib}^{\,b} + \vec{w}_g \tag{17}$$

Error Modeling Example

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{\,b} + \vec{nl}_a + \vec{w}_a \tag{16}$$

Biases

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{\,b} + G_g\vec{f}_{ib}^{\,b} + \vec{w}_g \tag{17}$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{\,b} + \vec{nl}_a + \vec{w}_a \quad (16)$$

Misalignment and SF Errors

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{\,b} + G_g\vec{f}_{ib}^{\,b} + \vec{w}_g \quad (17)$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{\,b} + \vec{nl}_a + \vec{w}_a \tag{16}$$

Non-linearity

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{\,b} + G_g\vec{f}_{ib}^{\,b} + \vec{w}_g \tag{17}$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{\,b} + \vec{nl}_a + \vec{w}_a \tag{16}$$

G-Sensitivity

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{\,b} + G_g\vec{f}_{ib}^{\,b} + \vec{w}_g \tag{17}$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{\,b} + \vec{nl}_a + \vec{w}_a \tag{16}$$

Noise

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{\,b} + G_g\vec{f}_{ib}^{\,b} + \vec{w}_g \tag{17}$$

Pos, Vel, Force and Angular Rate Errors

- Position error

$$\delta\vec{r}^{\gamma}_{\beta b} = \vec{r}^{\gamma}_{\beta b} - \hat{\vec{r}}^{\gamma}_{\beta b} \tag{18}$$

- Velocity error

$$\delta\vec{v}^{\gamma}_{\beta b} = \vec{v}^{\gamma}_{\beta b} - \hat{\vec{v}}^{\gamma}_{\beta b} \tag{19}$$

- Specific force errors

$$\delta\vec{f}^{b}_{ib} = \vec{f}^{b}_{ib} - \hat{\vec{f}}^{b}_{ib} \tag{20}$$

$$\Delta_e\vec{f}^{b}_{ib} = \Delta\vec{f}^{b}_{ib} - \Delta\hat{\vec{f}}^{b}_{ib} = -\delta\vec{f}^{b}_{ib} \tag{21}$$

- Angular rate errors

$$\delta\vec{\omega}^{b}_{ib} = \vec{\omega}^{b}_{ib} - \hat{\vec{\omega}}^{b}_{ib} \tag{22}$$

$$\Delta_e\vec{\omega}^{b}_{ib} = \Delta\vec{\omega}^{b}_{ib} - \Delta\hat{\vec{\omega}}^{b}_{ib} = -\delta\vec{\omega}^{b}_{ib} \tag{23}$$

Define

$$\delta C_b^\gamma = C_b^\gamma \hat{C}_\gamma^b = e^{[\delta\vec{\psi}^{\gamma}_{\gamma b}\times]} \approx \mathcal{I} + [\delta\vec{\psi}^{\gamma}_{\gamma b}\times] \tag{24}$$

This is the error in attitude resulting from errors in estimating the angular rates.

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])C_b^\gamma \tag{25}$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])\hat{C}_b^\gamma \tag{26}$$

Similarly we can attempt to estimate the specific force and angular rate by applying correction based on our estimate of the error.

$$\hat{\vec{f}}^{\,b}_{ib} = \tilde{\vec{f}}^{b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} \tag{27}$$

$$\hat{\vec{\omega}}^{b}_{ib} = \tilde{\vec{\omega}}^{b}_{ib} - \Delta\hat{\vec{\omega}}^{b}_{ib} \tag{28}$$

where $\hat{\vec{f}}^{\,b}_{ib}$ and $\hat{\vec{\omega}}^{b}_{ib}$ are the accelerometer and gyroscope estimated calibration values, respectively.