Lecture

Error Mechanization (ECEF)

EE 565: Position, Navigation and Timing

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Recall — ECEF Mechanization

- In continuous time notation
  - Attitude: $\dot{C}^e_b = C^e_b \Omega^b_{eb}$
  - Velocity: $\dot{\vec{v}}^{\,e}_{eb} = C^e_b \vec{f}^{\,b}_{ib} + \vec{g}^e_b - 2\Omega^i_{ie}\vec{v}^{\,e}_{eb}$
  - Position: $\dot{\vec{r}}^{\,e}_{ib} = \vec{v}^{\,e}_{eb}$
- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}^{\,e}_{eb} \\ \dot{\vec{v}}^{\,e}_{eb} \\ \dot{C}^i_b \end{bmatrix} = \begin{bmatrix} \vec{v}^{\,e}_{eb} \\ C^e_b \vec{f}^{\,b}_{ib} + \vec{g}^e_b - 2\Omega^i_{ie}\vec{v}^{\,e}_{eb} \\ C^e_b \Omega^b_{eb} \end{bmatrix} \tag{1}$$

where $\vec{\omega}^b_{ib} = \vec{\omega}^b_{ie} + \vec{\omega}^b_{eb}$, and $\Omega^b_{eb} = \Omega^b_{ib} - \Omega^b_{ie}$.

Question
What is the effect of sensor noise in the measurements $\vec{\omega}^b_{ib}$ and $\vec{f}^b_{ib}$ on the navigation solution position, velocity and attitude?

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1 Attitude

ECEF Attitude Error

$$\dot{C}^e_b = C^e_b\Omega^b_{eb} \;\; = C^e_b(\Omega^b_{ib} - \Omega^b_{ie}) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^e_{eb}\times])\hat{C}^e_b\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}^e_{eb}\times])\hat{C}^e_b\Omega^b_{eb} = [\delta\dot{\vec{\psi}}^e_{eb}\times]\hat{C}^e_b + (\mathcal{I} + [\delta\vec{\psi}^e_{eb}\times])\dot{\hat{C}}^e_b =$$

$$(\mathcal{I} + [\delta\vec{\psi}^e_{eb}\times])\hat{C}^e_b(\hat{\Omega}^b_{eb} + \delta\Omega^b_{ib} - \delta\Omega^b_{ie}) =$$

$$(\mathcal{I} + [\delta\vec{\psi}^e_{eb}\times])\hat{C}^e_b\hat{\Omega}^b_{eb} + \hat{C}^e_b(\delta\Omega^b_{ib} - \delta\Omega^b_{ie}) =$$

$$[\delta\dot{\vec{\psi}}^e_{eb}\times] = \hat{C}^e_b(\delta\Omega^b_{ib} - \delta\Omega^b_{ie})\hat{C}^b_e = [\hat{C}^e_b(\delta\vec{\omega}^b_{ib} - \delta\vec{\omega}^b_{ie})\times] \tag{2}$$

$$\delta\dot{\vec{\psi}}^e_{eb} = \hat{C}^e_b(\delta\omega^b_{ib} - \delta\vec{\omega}^b_{ie}) \tag{3}$$

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ECEF Attitude Error (cont.)

$$\begin{aligned}
\delta\dot{\vec{\psi}}^e_{eb} &= \hat{C}^e_b(\delta\vec{\omega}^b_{ib} - \delta\vec{\omega}^b_{ie}) \\
&= \hat{C}^e_b\delta\vec{\omega}^b_{ib} - \hat{C}^e_b(\vec{\omega}^b_{ie} - \hat{\vec{\omega}}^b_{ie}) \\
&= \hat{C}^e_b\delta\vec{\omega}^b_{ib} - (\hat{C}^e_b C^b_e - \mathcal{I})\vec{\omega}^e_{ie} \\
&= \hat{C}^e_b\delta\vec{\omega}^b_{ib} + \delta\vec{\psi}^e_{eb} \times \vec{\omega}^e_{ie}
\end{aligned}$$

$$\delta\dot{\vec{\psi}}^e_{eb} = \hat{C}^e_b\delta\vec{\omega}^b_{ib} - \vec{\Omega}^e_{ie}\delta\vec{\psi}^e_{eb} \tag{4}$$

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2 Velocity

Velocity

$$\dot{\vec{v}}^e_{eb} = C^e_b\vec{f}^b_{ib} + \vec{g}^e_b - 2\Omega^e_{ie}\vec{v}^e_{eb} \tag{5}$$

$$\begin{aligned}
\dot{\hat{\vec{v}}}^e_{eb} &= \hat{C}^e_b\hat{\vec{f}}^b_{ib} + \hat{\vec{g}}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^e_{eb} \\
&= (\mathcal{I} - [\delta\vec{\psi}^e_{eb}\times])C^e_b(\vec{f}^b_{ib} - \delta\vec{f}^b_{ib}) + \hat{\vec{g}}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^e_{eb}
\end{aligned} \tag{6}$$

$$\begin{aligned}
\delta\dot{\vec{v}}^e_{eb} = \dot{\vec{v}}^e_{eb} - \dot{\hat{\vec{v}}}^e_{eb} &= [\delta\vec{\psi}^e_{eb}\times]C^e_b\vec{f}^b_{ib} + \hat{C}^e_b\delta\vec{f}^b_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie}\delta\vec{v}^e_{eb} \\
&= [\delta\vec{\psi}^e_{eb}\times]\hat{C}^e_b\hat{\vec{f}}^b_{ib} + \hat{C}^e_b\delta\vec{f}^b_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie}\delta\vec{v}^e_{eb}
\end{aligned}$$

$$\delta\dot{\vec{v}}^e_{eb} = -[\hat{C}^e_b\hat{\vec{f}}^b_{ib}\times]\delta\vec{\psi}^e_{eb} + \hat{C}^e_b\delta\vec{f}^b_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie}\delta\vec{v}^e_{eb} \tag{7}$$

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3 Gravity

Gravity Error

Using Taylor series expansion, the gravity error as a function of position estimates and errors is derived to be

$$\delta\vec{g}^e_b \approx \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)}\frac{\hat{\vec{r}}^e_{eb}}{|\hat{\vec{r}}^e_{eb}|^2}(\hat{\vec{r}}^e_{eb})^T\delta\vec{r}^e_{eb} \tag{8}$$

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4 Position

Position

$$\dot{\vec{r}}^e_{eb} = \vec{v}^e_{eb} \tag{9}$$

$$\delta\dot{\vec{r}}^e_{eb} = \delta\vec{v}^e_{eb} \tag{10}$$

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5 Summary

Summary - in terms of $\delta \vec{f}^{\,b}_{ib}, \delta \vec{\omega}^{b}_{ib}$

$$\begin{pmatrix} \delta \dot{\vec{\psi}}^{e}_{eb} \\ \delta \dot{\vec{v}}^{\,e}_{eb} \\ \delta \dot{\vec{r}}^{\,e}_{eb} \end{pmatrix} = \begin{bmatrix} -\Omega^{e}_{ie} & 0_{3\times3} & 0_{3\times3} \\ -[\hat{C}^{e}_{b} \hat{\vec{f}}^{\,b}_{ib} \times] & -2\Omega^{e}_{ie} & \frac{2 g_0(\hat{L}_b)}{r^{e}_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^{e}_{eb}}{|\hat{\vec{r}}^{e}_{eb}|^2} (\hat{\vec{r}}^{e}_{eb})^T \\ 0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{pmatrix} \delta \vec{\psi}^{e}_{eb} \\ \delta \vec{v}^{\,e}_{eb} \\ \delta \vec{r}^{\,e}_{eb} \end{pmatrix} + \begin{bmatrix} 0 & \hat{C}^{e}_{b} \\ \hat{C}^{e}_{b} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \delta \vec{f}^{\,b}_{ib} \\ \delta \vec{\omega}^{b}_{ib} \end{pmatrix} \tag{11}$$

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A Basic Definitions

Notation Used

- *Truth value*
$$\vec{x}$$
- *Measured value*
$$\tilde{\vec{x}}$$
- *Estimated or computed value*
$$\hat{\vec{x}}$$
- *Error*
$$\delta \vec{x} = \vec{x} - \hat{\vec{x}}$$

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B Linearization

Linearization using Taylor Series Expansion

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta \vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta \dot{\vec{x}} = f(\hat{\vec{x}} + \delta \vec{x}, t) \tag{12}$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta \vec{x}, t) = \dot{\hat{\vec{x}}} + \delta \dot{\vec{x}} = &\ f(\hat{\vec{x}}, t) \ + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta \vec{x} + H.O.T \\ \approx &\ \dot{\hat{\vec{x}}} \ + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta \vec{x} \end{aligned}$$

$$\Rightarrow \delta \dot{\vec{x}} \approx \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta \vec{x} \tag{13}$$

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C Inertial Measurements

Actual Measurements

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}^b_{ib}$ and $\tilde{\vec{\omega}}^b_{ib}$, respectively, will be modeled as

$$\tilde{\vec{f}}^b_{ib} = \vec{f}^b_{ib} + \Delta\vec{f}^b_{ib} \tag{14}$$

$$\tilde{\vec{\omega}}^b_{ib} = \vec{\omega}^b_{ib} + \Delta\vec{\omega}^b_{ib} \tag{15}$$

where $\vec{f}^b_{ib}$ and $\vec{\omega}^b_{ib}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}^b_{ib}$ and $\Delta\vec{\omega}^b_{ib}$ represents the errors. In later lectures we will discuss more detailed description of these errors. .11

Error Modeling Example

Accelerometers

$$\tilde{\vec{f}}^b_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^b_{ib} + \vec{nl}_a + \vec{w}_a \tag{16}$$

Gyroscopes

$$\tilde{\vec{\omega}}^b_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^b_{ib} + G_g\vec{f}^b_{ib} + \vec{w}_g \tag{17}$$

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Pos, Vel, Force and Angular Rate Errors

- Position error

$$\delta\vec{r}^{\gamma}_{\beta b} = \vec{r}^{\gamma}_{\beta b} - \hat{\vec{r}}^{\gamma}_{\beta b} \tag{18}$$

- Velocity error

$$\delta\vec{v}^{\gamma}_{\beta b} = \vec{v}^{\gamma}_{\beta b} - \hat{\vec{v}}^{\gamma}_{\beta b} \tag{19}$$

- Specific force errors

$$\delta\vec{f}^b_{ib} = \vec{f}^b_{ib} - \hat{\vec{f}}^b_{ib} \tag{20}$$

$$\Delta_e\vec{f}^b_{ib} = \Delta\vec{f}^b_{ib} - \Delta\hat{\vec{f}}^b_{ib} = -\delta\vec{f}^b_{ib} \tag{21}$$

- Angular rate errors

$$\delta\vec{\omega}^b_{ib} = \vec{\omega}^b_{ib} - \hat{\vec{\omega}}^b_{ib} \tag{22}$$

$$\Delta_e\vec{\omega}^b_{ib} = \Delta\vec{\omega}^b_{ib} - \Delta\hat{\vec{\omega}}^b_{ib} = -\delta\vec{\omega}^b_{ib} \tag{23}$$

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D Attitude Error

Attitude Error Definition

Define

$$\delta C^{\gamma}_b = C^{\gamma}_b \hat{C}^b_{\gamma} = e^{[\delta\vec{\psi}^{\gamma}_{\gamma b}\times]} \approx \mathcal{I} + [\delta\vec{\psi}^{\gamma}_{\gamma b}\times] \tag{24}$$

This is the error in attitude resulting from errors in estimating the angular rates. .14

Attitude Error Properties

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])C_b^\gamma \tag{25}$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])\hat{C}_b^\gamma \tag{26}$$

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E Estimates of Sensor Measurements

Specific Force and Agnular Rates

Similarly we can attempt to estimate the specific force and angular rate by applying correction based on our estimate of the error.

$$\hat{\vec{f}}_{ib}^{b} = \tilde{\vec{f}}_{ib}^{b} - \Delta\hat{\vec{f}}_{ib}^{b} \tag{27}$$

$$\hat{\vec{\omega}}_{ib}^{b} = \tilde{\vec{\omega}}_{ib}^{b} - \Delta\hat{\vec{\omega}}_{ib}^{b} \tag{28}$$

where $\hat{\vec{f}}_{ib}^{b}$ and $\hat{\vec{\omega}}_{ib}^{b}$ are the accelerometer and gyroscope estimated calibration values, respectively. .16