

# Lecture

## Navigation Mathematics: Kinematics (Coordinate Frame Transformation)

EE 565: Position, Navigation and Timing

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### Coordinate Frame Transformation

- Determine the detailed kinematic relationships between the 4 major frames of interest
  - The Earth-Centered Inertial (ECI) coordinate frame (*i*-frame)
  - The Earth-Centered Earth-Fixed (ECEF) coordinate frame (*e*-frame)
  - The Local Navigation (Nav) coordinate frame (*n*-frame)
  - The Body coordinate frame (*b*-frame)

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### ECI/ECEF

- Relationship between the ECI and ECEF frames
  - ECI & ECEF have co-located origins

$$\vec{r}_{ie} = \dot{\vec{r}}_{ie} = \ddot{\vec{r}}_{ie} = 0$$

- The *x*, *y*, and *z* axis of the ECI & ECEF frames are coincident at time  $t_0$
- The ECEF frame rotates about the common *z*-axis at a fixed rate ( $\omega_{ie}$ )
  - \* Ignoring minor speed variations (precession & nutation)  $\omega_{ie} = 72.921151467 \mu\text{rad/sec}$  (WGS84) which is  $\approx 15^\circ/\text{hr}$

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### ECI/ECEF

- The angular velocity and acceleration are

$$\vec{\omega}_{ie}^i = \begin{bmatrix} 0 \\ 0 \\ \omega_{ie} \end{bmatrix} \quad \dot{\vec{\omega}}_{ie}^i = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- The angle of rotation is

$$\begin{aligned} \theta_{ie} &= \omega_{ie}(t - t_0) \\ &= \omega_{ie}t + \theta_{GMST} \end{aligned}$$

where **GMST** is the Greenwich mean sidereal time

- The orientation of frame  $\{e\}$  wrt frame  $\{i\}$  becomes

$$C_e^i = R_{(z, \theta_{ie})} = \begin{bmatrix} \cos \theta_{ie} & -\sin \theta_{ie} & 0 \\ \sin \theta_{ie} & \cos \theta_{ie} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note:  $\vec{\omega}_{ie}^i = \vec{\omega}_{ie}^e$

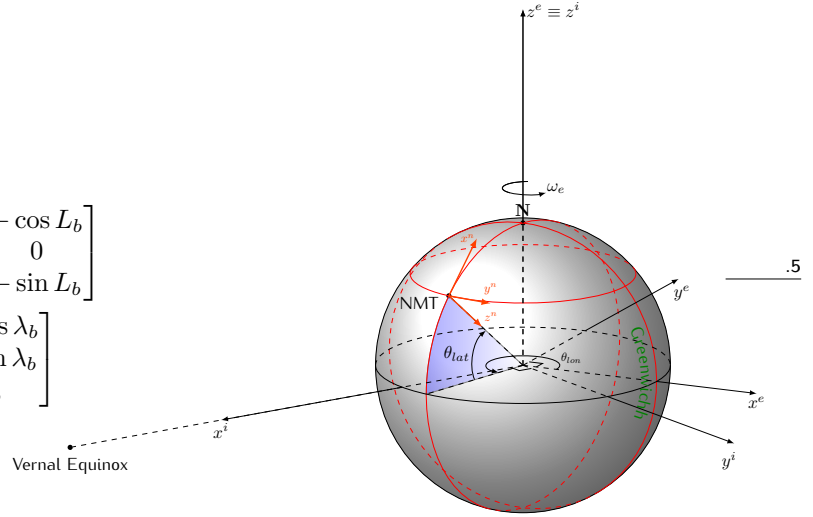
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## ECEF/Nav

- Description of the navigation frame
  - Orientation of the  $n$ -frame wrt the  $e$ -frame

$$\begin{aligned} C_n^e &= R_{(z, \lambda_b)} R_{(\bar{y}, -L_b - 90^\circ)} \\ &= \begin{bmatrix} \cos \lambda_b & -\sin \lambda_b & 0 \\ \sin \lambda_b & \cos \lambda_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b & 0 & -\cos L_b \\ 0 & 1 & 0 \\ \cos L_b & 0 & -\sin L_b \end{bmatrix} \\ &= \begin{bmatrix} -\sin L_b \cos \lambda_b & -\sin \lambda_b & -\cos L_b \cos \lambda_b \\ -\sin L_b \sin \lambda_b & \cos \lambda_b & -\cos L_b \sin \lambda_b \\ \cos L_b & 0 & -\sin L_b \end{bmatrix} \end{aligned}$$

where geodetic Lat =  $L_b$  and Geodetic Lon =  $\lambda_b$



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## ECEF/Nav

- Angular velocity of the  $n$ -frame wrt the  $e$ -frame resolved in the  $e$ -frame as a skew-symmetric matrix

$$\dot{C}_n^e = C_n^e \Omega_{en}^n = \Omega_{en}^e C_n^e$$

$$\begin{aligned} \Omega_{en}^e &= \dot{C}_n^e [C_n^e]^T \\ &= \begin{bmatrix} s_{L_b} s_{\lambda_b} \dot{\lambda}_b - c_{L_b} c_{\lambda_b} \dot{L}_b & -c_{\lambda_b} \dot{\lambda}_b & -c_{\lambda_b} s_{L_b} \dot{L}_b + c_{L_b} s_{\lambda_b} \dot{\lambda}_b \\ -c_{L_b} s_{\lambda_b} \dot{L}_b - c_{\lambda_b} s_{L_b} \dot{\lambda}_b & -s_{\lambda_b} \dot{\lambda}_b & s_{L_b} s_{\lambda_b} \dot{L}_b - c_{L_b} c_{\lambda_b} \dot{\lambda}_b \\ -s_{L_b} \dot{L}_b & 0 & -c_{L_b} \dot{L}_b \end{bmatrix} [C_n^e]^T \\ &= \begin{bmatrix} 0 & -\dot{\lambda}_b & -\dot{L}_b \cos(\lambda_b) \\ \dot{\lambda}_b & 0 & -\dot{L}_b \sin(\lambda_b) \\ \dot{L}_b \cos(\lambda_b) & \dot{L}_b \sin(\lambda_b) & 0 \end{bmatrix} \end{aligned}$$

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## ECEF/Nav

- The angular velocity vector

$$\vec{\omega}_{en}^e = \begin{bmatrix} \sin(\lambda) \dot{L}_b \\ -\cos(\lambda) \dot{L}_b \\ \dot{\lambda}_b \end{bmatrix} \quad \vec{\omega}_{en}^n = [C_n^e]^T \vec{\omega}_{en}^e = \begin{bmatrix} \cos(\lambda) \dot{\lambda}_b \\ -\dot{L}_b \\ -\sin(L_b) \dot{\lambda}_b \end{bmatrix}$$

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## ECI/ECEF/Nav

- Hence the orientation of the  $n$ -frame wrt the  $i$ -frame becomes

$$C_n^i = C_e^i C_n^e = \begin{bmatrix} c_{\theta_{ie}} & -s_{\theta_{ie}} & 0 \\ s_{\theta_{ie}} & c_{\theta_{ie}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_{L_b} c_{\lambda_b} & -s_{\lambda_b} & -c_{L_b} c_{\lambda_b} \\ -s_{L_b} s_{\lambda_b} & c_{\lambda_b} & -c_{L_b} s_{\lambda_b} \\ c_{L_b} & 0 & -s_{L_b} \end{bmatrix}$$

$$= \begin{bmatrix} -\sin(L_b) \cos(\theta_{ie} + \lambda_b) & -\sin(\theta_{ie} + \lambda_b) & -\cos(L_b) \cos(\theta_{ie} + \lambda_b) \\ -\sin(L_b) \sin(\theta_{ie} + \lambda_b) & \cos(\theta_{ie} + \lambda_b) & -\cos(L_b) \sin(\theta_{ie} + \lambda_b) \\ \cos(L_b) & 0 & -\sin(L_b) \end{bmatrix}$$

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## ECI/ECEF/Nav

- The angular velocity of the  $n$ -frame wrt the  $i$ -frame resolved in the  $i$ -frame is

$$\vec{\omega}_{in}^i = \vec{\omega}_{ie}^i + C_e^i \vec{\omega}_{en}^e$$

$$= \begin{bmatrix} 0 \\ 0 \\ \omega_{ie} \end{bmatrix} + \begin{bmatrix} \cos(\theta_{ie}) & -\sin(\theta_{ie}) & 0 \\ \sin(\theta_{ie}) & \cos(\theta_{ie}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(\lambda_b) \dot{L}_b \\ -\cos(\lambda_b) \dot{L}_b \\ \dot{\lambda}_b \end{bmatrix}$$

$$= \begin{bmatrix} \sin(\theta_{ie} + \lambda_b) \dot{L}_b \\ -\cos(\theta_{ie} + \lambda_b) \dot{L}_b \\ \omega_{ie} + \dot{\lambda}_b \end{bmatrix}$$

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## ECEF/Nav

- The vector from the origin of the  $e$ -frame to the  $n$ -frame origin resolved in the  $e$ -frame (from the last lecture)

$$\vec{r}_{eb}^e = \begin{bmatrix} (R_E + h_b) \cos(L_b) \cos(\lambda_b) \\ (R_E + h_b) \cos(L_b) \sin(\lambda_b) \\ (R_E(1 - e^2) + h_b) \sin(L_b) \end{bmatrix} = \vec{r}_{en}^e$$

Origins of the  $n$ -frame and the  $b$ -frame are the same

- The velocity of the  $n$ -frame wrt the  $e$ -frame resolved in the  $e$ -frame

$$\vec{v}_{en}^e = \frac{d}{dt} \vec{r}_{en}^e = \frac{\partial \vec{r}_{en}^e}{\partial L_b} \dot{L}_b + \frac{\partial \vec{r}_{en}^e}{\partial \lambda_b} \dot{\lambda}_b + \frac{\partial \vec{r}_{en}^e}{\partial h_b} \dot{h}_b$$

$$= \begin{bmatrix} -\sin(L_b) \cos(\lambda_b) & -\sin(\lambda_b) & -\cos(L_b) \cos(\lambda_b) \\ -\sin(L_b) \sin(\lambda_b) & \cos(\lambda_b) & -\cos(L_b) \sin(\lambda_b) \\ \cos(L_b) & 0 & -\sin(L_b) \end{bmatrix} \begin{bmatrix} (R_N + h_b) \dot{L}_b \\ \cos(L_b) (R_E + h_b) \dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix}$$

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## ECEF/Nav

- Recalling the form of  $C_n^e$  suggests that

$$\vec{v}_{en}^e = C_n^e \begin{bmatrix} (R_N + h_b) \dot{L}_b \\ \cos(L_b) (R_E + h_b) \dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix} = C_n^e \vec{v}_{en}^n$$

- and hence,

$$\vec{v}_{en}^n = \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix}$$

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## ECEF/Nav

- Restating  $\vec{v}_{en}^n$  as

$$\vec{v}_{en}^n = \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix} = \begin{bmatrix} v_{en,N}^n \\ v_{en,E}^n \\ v_{en,D}^n \end{bmatrix}$$

- and recalling that

$$\vec{\omega}_{en}^n = \begin{bmatrix} \cos(L_b)\dot{\lambda}_b \\ -\dot{L}_b \\ -\sin(L_b)\dot{\lambda}_b \end{bmatrix}$$

- suggests that

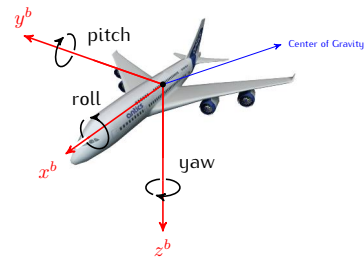
$$\vec{\omega}_{en}^n = \begin{bmatrix} \vec{v}_{en,E}^n / (R_E + h_b) \\ -\vec{v}_{en,N}^n / (R_N + h_b) \\ -\tan(L_b)\vec{v}_{en,E}^n / (R_E + h_b) \end{bmatrix}$$

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## Body Frame

- Description wrt the body frame
  - Orientation of the  $b$ -frame wrt the  $n$ -frame in terms of relative yaw ( $\psi$ ), pitch ( $\theta$ ), then roll ( $\phi$ ) angles

$$\begin{aligned} C_b^n &= R_{(\vec{z},\psi)} R_{(\vec{y},\theta)} R_{(\vec{x},\phi)} = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix} \\ &= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \end{aligned}$$



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## Body Frame

- The angular velocity of the  $b$ -frame wrt the  $i$ -frame resolved/coordinated in the  $i$ -frame

$$\begin{aligned} \vec{\omega}_{ib}^i &= \vec{\omega}_{in}^i + C_n^i \vec{\omega}_{nb}^n \\ &= \vec{\omega}_{ie}^i + C_e^i \vec{\omega}_{en}^e + C_n^i \vec{\omega}_{nb}^n \end{aligned}$$

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## Body Frame

- Position vectors to the origin of the body frame
  - The origins of the body and Nav frames are co-incident

$$\vec{r}_{nb} = \vec{0}$$

- The origins of the ECI and ECEF frames are co-incident

$$\vec{r}_{eb} = \vec{r}_{ib} = \vec{r}_{en} = \vec{r}_{in}$$

- Velocity of the  $b$ -frame wrt the  $i$ -frame resolved in the  $i$ -frame
  - \* A moving point in a rotation frame

$$\begin{aligned}\vec{v}_{ib}^i &= \frac{d}{dt} \vec{r}_{ib}^i = \frac{d}{dt} C_e^i \vec{r}_{eb}^e \\ &= C_e^i \Omega_{ie}^e \vec{r}_{eb}^e + C_e^i \vec{v}_{eb}^e \\ &= C_e^i (\Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e)\end{aligned}$$

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## Body Frame

- Acceleration of the  $b$ -frame wrt the  $i$ -frame resolved in the  $i$ -frame
  - A moving point in a rotation frame

$$\begin{aligned}\vec{a}_{ib}^i &= \frac{d}{dt} \vec{v}_{ib}^i = \frac{d}{dt} (C_e^i (\Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e)) \\ &= \dot{C}_e^i (\Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e) + C_e^i \left( \overset{\dot{\omega}=0}{\cancel{\dot{\Omega}_{ie}^e}} \vec{r}_{eb}^e + \Omega_{ie}^e \dot{\vec{r}}_{eb}^e + \dot{\vec{v}}_{eb}^e \right) \\ &= C_e^i \Omega_{ie}^e (\Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e) + C_e^i (\Omega_{ie}^e \vec{v}_{eb}^e + \vec{a}_{eb}^e) \\ &= C_e^i (\Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e + 2\Omega_{ie}^e \vec{v}_{eb}^e + \vec{a}_{eb}^e)\end{aligned}$$

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