

# EE 565: Position, Navigation and Timing

## Navigation Mathematics: Kinematics (Earth Surface & Gravity Models)

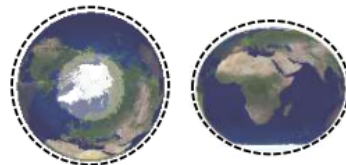
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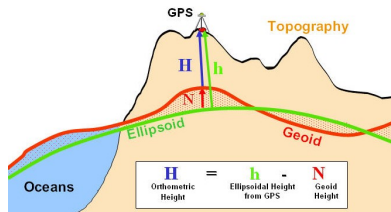
February 13, 2020

- The earth can be modeled as an oblate spheroid
  - A circular cross section when viewed from the polar axis (top view)
  - An elliptical cross-section when viewed perpendicular to the polar axis (side view)



Ratio exaggerated

- This ellipsoid (i.e., oblate spheroid) is an approximation of the “geoid”
- The geoid is a gravitational equipotential surface which “best” fits (in the least square sense) the mean sea level



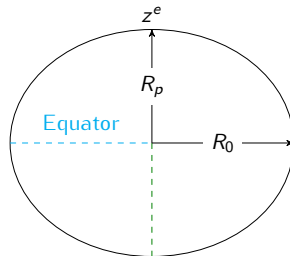
[www.nrcan.gc.ca](http://www.nrcan.gc.ca)

- WGS 84 provides as model of the earth's geoid
  - More recently replace by EGM 2008
- The equatorial radius radius  $R_0 = 6,378,137.0\text{m}$
- The polar radius radius  $R_p = 6,356,752.3142\text{m}$
- Eccentricity of the ellipsoid

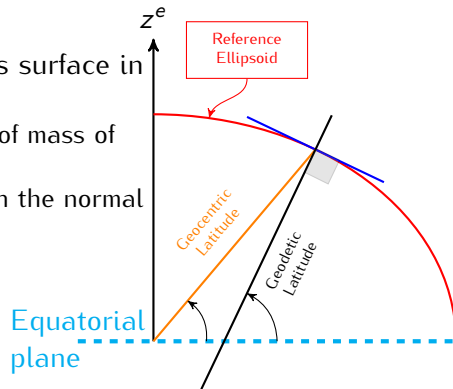
$$e = \sqrt{1 - \frac{R_p^2}{R_0^2}} \approx 0.0818$$

- Flattening of the ellipsoid

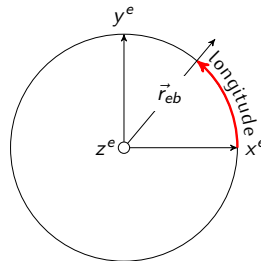
$$f = \frac{R_0 - R_p}{R_0} \approx \frac{1}{298}$$



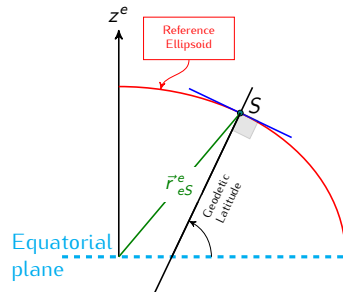
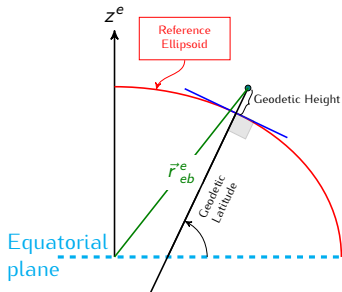
- We can define a position “near” the earth’s surface in terms of latitude, longitude, and height
  - Geocentric latitude intersects the center of mass of the earth
  - Geodetic latitude ( $L$ ) is the angle between the normal to the ellipsoid and the equatorial plane



- The longitude ( $\lambda$ ) is the angle from the x-axis of the ECEF frame to the projection of  $\vec{r}_{eb}$  onto the equatorial plane



- The geocentric radius is the distance from center of the Earth to the point  $S$
- The geodetic (or ellipsoidal) height ( $h$ ) is the distance along the normal from the ellipsoid to the body

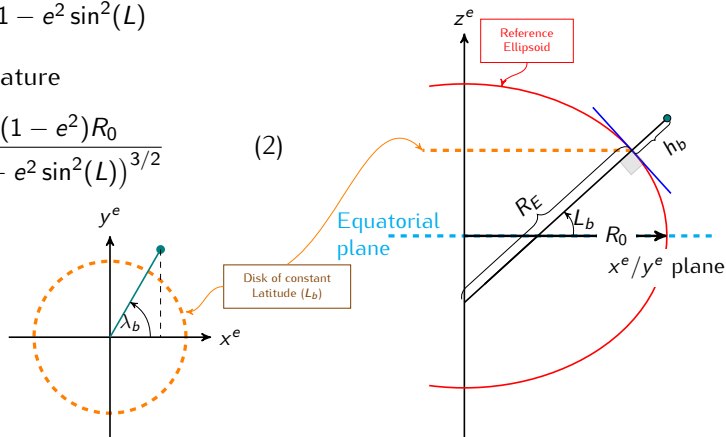


- Transverse radius of curvature

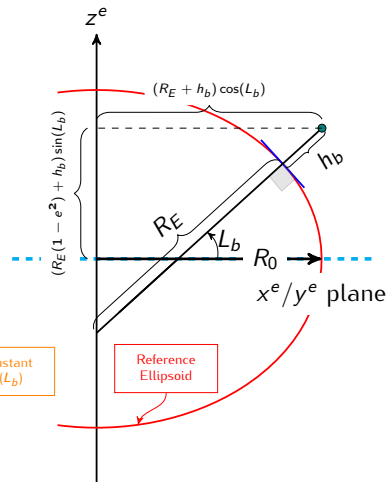
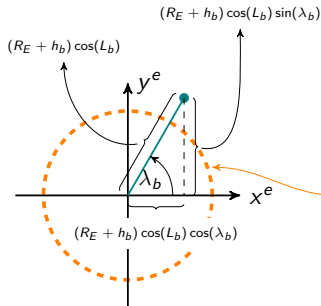
$$R_E(L) = \frac{R_0}{\sqrt{1 - e^2 \sin^2(L)}} \quad (1)$$

- Meridian radius of curvature

$$R_N(L) = \frac{(1 - e^2)R_0}{(1 - e^2 \sin^2(L))^{3/2}} \quad (2)$$



$$\vec{r}_{eb}^e = \begin{bmatrix} x_{eb}^e \\ y_{eb}^e \\ z_{eb}^e \end{bmatrix} = \begin{bmatrix} (R_E + h_b) \cos(L_b) \cos(\lambda_b) \\ (R_E + h_b) \cos(L_b) \sin(\lambda_b) \\ (R_E(1 - e^2) + h_b) \sin(L_b) \end{bmatrix}$$





- Specific force ( $\vec{f}_{ib}$ )
  - Non-gravitational force per unit mass (unit of acceleration)
    - Accelerometers measure specific force
- Specific force sensed when stationary (*wrt* earth) is referred to as the acceleration due to gravity ( $\vec{g}_b$ )
  - Actually, the reaction to this force
- Gravitational force ( $\gamma_{ib}$ ) is result of mass attraction
  - The gravitational mass attraction force is different from the acceleration due to gravity

- Relationship between specific force, inertial acceleration, and gravitational attraction

## Specific force

$$\vec{f}_{ib} = \vec{a}_{ib} - \vec{\gamma}_{ib} \quad (3)$$

- When stationary on the surface of the earth
  - A fixed point in a rotating frame

$$\ddot{\vec{r}}_{02}^0(t) = \dot{\vec{\omega}}_{01}^0 \times \vec{r}_{12}^0(t) + \vec{\omega}_{01}^0 \times (\vec{\omega}_{01}^0 \times \vec{r}_{12}^0(t))$$

$\dot{\omega} = 0$

- Consider frame {0} to be the {i} frame, {1}={e}, and {2}={b} gives

$$\ddot{\vec{r}}_{ib}^i(t) = \vec{\omega}_{ie}^i \times (\vec{\omega}_{ie}^i \times \vec{r}_{eb}^i(t))$$

- coordinatizing in the e-frame

$$\ddot{\vec{r}}_{ib}^e(t) = \vec{\omega}_{ie}^e \times (\vec{\omega}_{ie}^e \times \vec{r}_{eb}^e(t))$$

- Thus, when stationary on the surface of the earth the acceleration is due to centrifugal force

$$\vec{a}_{ib}^e = \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e$$

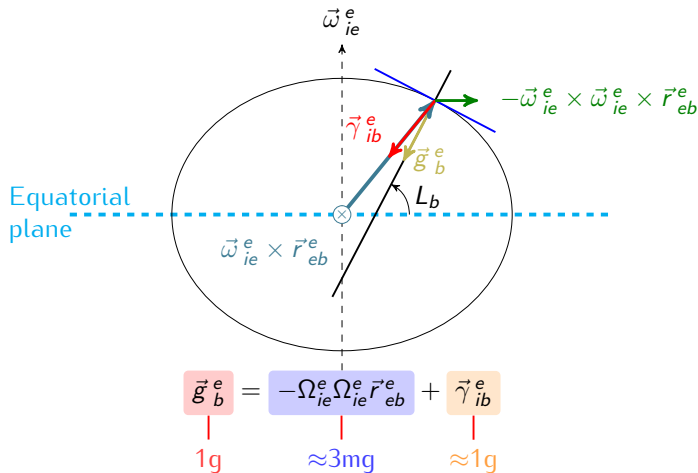
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$$\vec{a}_{ib}^e = \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e$$

- Therefore, the acceleration due to gravity is

$$\vec{g}_b^e = - \left. \vec{f}_{ib} \right|_{\vec{v}_{eb}^e=0} = -\Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e + \vec{\gamma}_{ib}^e \quad (4)$$



- Now,  $\vec{\omega}_{ie}^e = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega_{ie}$  and hence,  $\Omega_{ie}^e = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_{ie}$ , and thus

$$\vec{g}_b^e = \vec{\gamma}_{ib}^e + \omega_{ie}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{r}_{eb}^e$$

- The WGS 84 model of acceleration due to gravity (on the ellipsoid) can be approximated by (Somigliana model)

$$g_0(L_b) = 9.7803253359 \frac{(1 + 0.001931853 \sin^2(L))}{\sqrt{1 - e^2 \sin^2(L)}} \quad (5)$$

$$g_{b,D}^n = g_0(L_b, h_b) \left\{ 1 - \frac{2}{R_0} \left[ 1 + f(1 - 2 \sin^2 L_b) + \frac{\omega_{ie}^2 R_0^2 R_p}{\mu} \right] h_b + \frac{3}{R_0^2} h_b^2 \right\} \quad (6)$$

where  $\mu = 3.986004418 \times 10^{14} \text{ m}^3/\text{s}^2$  is the WGS 84 Earth's gravitational constant.

- On March 17, 2002 NASA launched the Gravity Recovery and Climate Experiment (GRACE) which led to the development of some of the most precise Earth gravity models.

