Lecture

Navigation Mathematics: Angular and Linear Velocity

EE 565: Position, Navigation and Timing

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Lecture Topics

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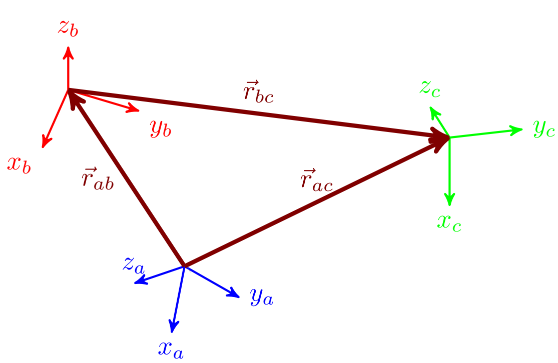
1 Review

Review



- translation between frames $\{a\}$ and $\{c\}$:

$$\vec{r}_{ac} = \vec{r}_{ab} + \vec{r}_{bc}$$

- written *wrt*/frame $\{a\}$

$$\begin{aligned}\vec{r}^{\,a}_{ac} &= \vec{r}^{\,a}_{ab} + \vec{r}^{\,a}_{bc} \\ &= \vec{r}^{\,a}_{ab} + C^a_b \vec{r}^{\,b}_{bc}\end{aligned}$$

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2 Introduction to Velocity

Introduction to Velocity

- Given relationship for translation between moving (rotating and translating) frames

$$\vec{r}^{\,a}_{ac} = \vec{r}^{\,a}_{ab} + C^a_b \vec{r}^{\,b}_{bc}$$

what is linear velocity between frames?

$$\begin{aligned} \dot{\vec{r}}^{a}_{ac} &\equiv \frac{d}{dt}\vec{r}^{\,a}_{ac} \\ &= \frac{d}{dt}\left(\vec{r}^{\,a}_{ab} + C^a_b \vec{r}^{\,b}_{bc}\right) \\ &= \dot{\vec{r}}^{a}_{ab} + \dot{C}^a_b \vec{r}^{\,b}_{bc} + C^a_b \dot{\vec{r}}^{b}_{bc} \end{aligned}$$

- Why is $\dot{C}^a_b \neq 0$ in general? Recoordinatization of $\vec{r}^{\,b}_{bc}$ is time-dependent.
- $\dot{C}^a_b$ is directly related to angular velocity between frames $\{a\}$ and $\{b\}$.

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3 Derivative of Rotation Matrix and Angular Velocity - Approach I

First approach to $\frac{d}{dt}C$ and angular velocity

Given a rotation matrix C, one of its properties is

$$[C^a_b]^T C^a_b = C^a_b [C^a_b]^T = \mathcal{I}$$

Taking the time-derivative of the "right-inverse" property

$$\frac{d}{dt}\left(C^a_b [C^a_b]^T\right) = \frac{d}{dt}\mathcal{I}$$

$$\Rightarrow \underbrace{\dot{C}^a_b [C^a_b]^T}_{\Omega^a_{ab}} + \underbrace{\underbrace{C^a_b [\dot{C}^a_b]^T}_{(\dot{C}^a_b [C^a_b]^T)^T}}_{[\Omega^a_{ab}]^T} = 0$$

$$\Rightarrow \Omega^a_{ab} + [\Omega^a_{ab}]^T = 0$$

$$\Rightarrow \Omega^a_{ab} \text{ is skew-symmetric!}$$

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First approach to $\frac{d}{dt}C$ and angular velocity

Define this skew-symmetric matrix Ω^a_{ab}

$$\Omega^a_{ab} = [\vec{\omega}^{\,a}_{ab} \times] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Note $\Omega^a_{ab} = \dot{C}^a_b [C^a_b]^T$

$$\Rightarrow \dot{C}^a_b = \Omega^a_{ab} C^a_b$$

is a means of finding derivative of rotation matrix provided we can further understand Ω^a_{ab}.

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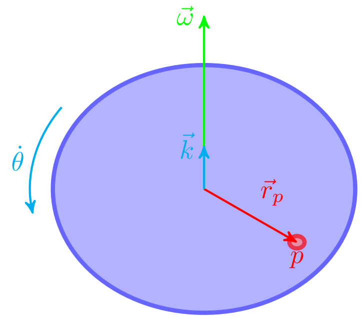
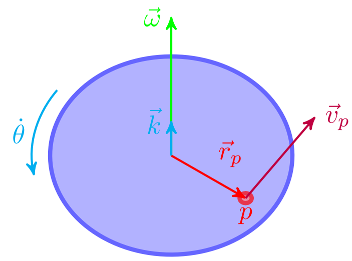
First approach to $\frac{d}{dt}C$ and angular velocity

Now for some insight into physical meaning of Ω^a_{ab}.

- Consider a point p on a rigid body rotating with angular velocity $\vec{\omega} = [\omega_x,\ \omega_y,\ \omega_z]^T = \dot{\theta}\vec{k} = \dot{\theta}[k_x,\ k_y,\ k_z]^T$ with $\vec{k}$ a unit vector.



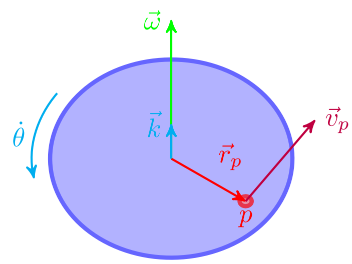
First approach to $\frac{d}{dt}C$ and angular velocity



From mechanics, linear velocity $\vec{v}_p$ of point is

$$\vec{v}_p = \vec{\omega} \times \vec{r}_p = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \omega_y r_z - \omega_z r_y \\ \omega_z r_x - \omega_x r_z \\ \omega_x r_y - \omega_y r_x \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}_{?} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

First approach to $\frac{d}{dt}C$ and angular velocity



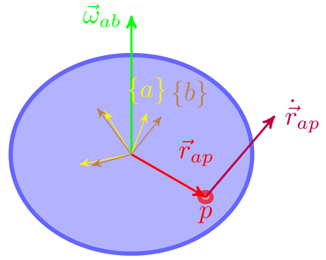
From mechanics, linear velocity $\vec{v}_p$ of point is

$$\vec{v}_p = \vec{\omega} \times \vec{r}_p = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \omega_y r_z - \omega_z r_y \\ \omega_z r_x - \omega_x r_z \\ \omega_x r_y - \omega_y r_x \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}_{\Omega = [\vec{\omega}\times]} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$\Rightarrow$ Ω represents angular velocity and performs cross product

First approach to $\frac{d}{dt}C$ and angular velocity

Now let's add fixed frame $\{a\}$ and rotating frame $\{b\}$ attached to moving body such that there is angular velocity $\vec{\omega}_{ab}$ between them.



Start with position

$$\vec{r}^{\,a}_{ap} = \underbrace{\vec{r}^{\,a}_{ab}}_{0} + C^a_b \vec{r}^{\,b}_{bp}$$

and take derivative *wrt* time

$$\begin{aligned} \dot{\vec{r}}^{a}_{ap} &= \underbrace{\dot{C}^a_b}_{\Omega^a_{ab} C^a_b} \vec{r}^{\,b}_{bp} + \underbrace{C^a_b \dot{\vec{r}}^{b}_{bp}}_{0} \\ &= \Omega^a_{ab} C^a_b \vec{r}^{\,b}_{bp} \\ &= \Omega^a_{ab} \vec{r}^{\,a}_{bp} = [\vec{\omega}^{\,a}_{ab} \times] \vec{r}^{\,a}_{bp} \end{aligned}$$

from which it is observed (compare to $\vec{v}_p = \vec{\omega} \times \vec{r}_p$) that Ω^a_{ab} represents cross product with angular velocity $\vec{\omega}^{\,a}_{ab}$.

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4 Derivative of Rotation Matrix and Angular Velocity - Approach II

Second approach to $\frac{d}{dt}C$ and angular velocity

- Another approach to developing derivative of rotation matrix and angular velocity is based upon angle-axis representation of orientation and rotation matrix as exponential.
- This approach is included in notes.

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Second approach to $\frac{d}{dt}C$ and angular velocity

- Since the relative and fixed axis rotations must be performed in a particular order, their derivatives are somewhat challenging
- The angle-axis format, however, is readily differentiable as we can encode the 3 parameters by

$$\vec{K} \equiv \vec{k}(t)\theta(t) = \begin{bmatrix} K_1(t) \\ K_2(t) \\ K_3(t) \end{bmatrix}$$

  where $\theta = \|\vec{K}\|$
- Hence,

$$\frac{d}{dt}\vec{K}(t) = \begin{bmatrix} \dot{K}_1(t) \\ \dot{K}_2(t) \\ \dot{K}_3(t) \end{bmatrix}$$

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Second approach to $\frac{d}{dt}C$ and angular velocity

- For a sufficiently "small" time interval we can often consider the axis of rotation to be $\approx$ constant (i.e., $\vec{k}(t) = \vec{k}$)

$$\begin{aligned} \frac{d}{dt}\vec{K}(t) &\approx \frac{d}{dt}\left(\vec{k}\theta(t)\right) \\ &= \vec{k}\dot{\theta}(t) \end{aligned}$$

- This is referred to as the angular velocity ($\vec{\omega}(t)$) or the so called "body reference" angular velocity

  Angular Velocity

$$\vec{\omega}(t) \equiv \vec{k}\dot{\theta}(t)$$

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Second approach to $\frac{d}{dt}C$ and angular velocity

- This definition of the angular velocity can also be related back to the rotation matrix. Recalling that

$$C_b^a(t) = R_{\vec{k}\,^a_{ab},\theta(t)} = e^{\kappa^a_{ab}\theta(t)}$$

- Hence,

$$\begin{aligned}\frac{d}{dt}C_b^a(t) &= \frac{d}{dt}e^{\kappa^a_{ab}\theta(t)} \\ &= \frac{\partial e^{\kappa^a_{ab}\theta(t)}}{\partial\theta}\frac{d\theta}{dt} \\ &= \kappa^a_{ab}e^{\kappa^a_{ab}\theta(t)}\dot{\theta}(t) \\ &= \left(\kappa^a_{ab}\dot{\theta}(t)\right)C_b^a(t)\end{aligned}$$

$$\Rightarrow \dot{C}_b^a(t)\left[C_b^a(t)\right]^T = \kappa^a_{ab}\dot{\theta}(t)$$

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Second approach to $\frac{d}{dt}C$ and angular velocity

Notice that

$$\begin{aligned}\kappa^a_{ab}\dot{\theta}(t) &= Skew\left[k^a_{ab}\right]\dot{\theta}(t) \\ &= Skew\left[k^a_{ab}\dot{\theta}(t)\right] \\ &= Skew\left[\vec{\omega}\,^a_{ab}\right] = \Omega^a_{ab}\end{aligned}$$

Therefore,

$$\dot{C}_b^a(t)\left[C_b^a(t)\right]^T = \Omega^a_{ab}$$

or

$$\dot{C}_b^a = \Omega^a_{ab}C_b^a$$

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Second approach to $\frac{d}{dt}C$ and angular velocity

Note

$$\kappa\vec{a} = \begin{bmatrix}0 & -k_3 & k_2\\ k_3 & 0 & -k_1\\ -k_2 & k_1 & 0\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix} = \begin{bmatrix}k_2a_3 - k_3a_2\\ k_3a_1 - k_1a_3\\ k_1a_2 - k_2a_1\end{bmatrix} = \vec{k}\times\vec{a}$$

Hence, we can think of the skew-symmetric matrix as

$$\kappa = [\vec{k}\times]$$

or, in the case of angular velocity

$$\Omega = [\vec{\omega}\times]$$

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5 Properties of Skew-symmetric Matrices

Properties of Skew-symmetric Matrices

$$\begin{aligned} C\Omega C^T\vec{b} &= C\left[\vec{\omega} \times \left(C^T\vec{b}\right)\right] \\ &= C\vec{\omega} \times \left(CC^T\vec{b}\right) \\ &= C\vec{\omega} \times \vec{b} \\ &= [C\vec{\omega}\times]\vec{b} \end{aligned}$$

Therefore (from above),

$$C\Omega C^T = C[\vec{\omega}\times]C^T = [C\vec{\omega}\times]$$

and (via distributive property)

$$C[\vec{\omega}\times] = [C\vec{\omega}\times]C$$

noting both $\vec{\omega}$ and vector with which cross-product will be taken are assumed to be in the same coordinate frame and thus both need to be recoordinatized. .17

Properties of Skew-symmetric Matrices

$$\begin{aligned} \dot{C}_b^a &= \Omega_{ab}^a C_b^a \\ &= [\vec{\omega}\,_{ab}^{a}\times]C_b^a \\ &= [C_b^a\vec{\omega}\,_{ab}^{b}\times]C_b^a \\ &= C_b^a[\vec{\omega}\,_{ab}^{b}\times] \\ &= C_b^a\Omega_{ab}^b \end{aligned}$$

$$\Rightarrow \dot{C}_b^a = \Omega_{ab}^a C_b^a = C_b^a\Omega_{ab}^b$$

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Summary of Angular Velocity and Notation

Angular velocity can be

- described as a vector
  - the angular velocity of the b-frame *wrt* the a-frame resolved in the c-frame, $\vec{\omega}\,_{ab}^{c}$
  - $\vec{\omega}_{ab} = -\vec{\omega}_{ba}$
- described as a skew-symmetric matrix $\Omega_{ab}^c = [\vec{\omega}\,_{ab}^{c}\times]$
  - the skew-symmetric matrix is equivalent to the vector cross product when pre-multiplying another vector
- related to the derivative of the rotation matrix

$$\begin{aligned} \dot{C}_b^a &= \Omega_{ab}^a C_b^a = C_b^a\Omega_{ab}^b \\ \dot{C}_b^a &= -\Omega_{ba}^a C_b^a = -C_b^a\Omega_{ba}^b \end{aligned}$$

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6 Propagation/Addition of Angular Velocity

Propagation/Addition of Angular Velocity

Consider the derivative of the composition of rotations $C_2^0 = C_1^0 C_2^1$.

$$
\begin{aligned}
& \tfrac{d}{dt} C_2^0 && = \frac{d}{dt} C_1^0 C_2^1 \\
\Rightarrow \quad & \dot{C}_2^0 && = \dot{C}_1^0 C_2^1 + C_1^0 \dot{C}_2^1 \\
\Rightarrow \quad & \Omega_{02}^0 C_2^0 && = \Omega_{01}^0 C_1^0 C_2^1 + C_1^0 C_2^1 \Omega_{12}^2 \\
\Rightarrow \quad & \Omega_{02}^0 && = \Omega_{01}^0 C_2^0 \left[C_2^0\right]^T + C_2^0 \Omega_{12}^2 \left[C_2^0\right]^T \\
\Rightarrow \quad & [\vec{\omega}^{\,0}_{02} \times] && = [\vec{\omega}^{\,0}_{01} \times] + [C_2^0 \vec{\omega}^{\,2}_{12} \times] \\
\Rightarrow \quad & \vec{\omega}^{\,0}_{02} && = \vec{\omega}^{\,0}_{01} + \vec{\omega}^{\,0}_{12}
\end{aligned}
$$

$\Rightarrow$ angular velocities (as vectors) add so long as resolved common coordinate system .20

7 Linear Position, Velocity and Acceleration

Linear Position

Consider the motion of a fixed point (origin of frame $\{2\}$) in a rotating frame (frame $\{1\}$) as seen from an inertial (frame $\{0\}$)

- frames $\{0\}$ and $\{1\}$ have the same origin
- frame $\{1\}$ rotates (about a unit vector $\vec{k}$) *wrt* frame $\{0\}$
- origin of frame $\{2\}$ is fixed *wrt* frame $\{1\}$

Position:

$$
\begin{aligned}
\vec{r}^{\,0}_{02}(t) &= \cancelto{\vec{0}}{\vec{r}^{\,0}_{01}(t)} + \vec{r}^{\,0}_{12}(t) \\
&= C_1^0(t) \vec{r}^{\,1}_{12}
\end{aligned}
$$

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Linear Velocity

Linear velocity:

$$
\begin{aligned}
\dot{\vec{r}}^{\,0}_{02}(t) &= \frac{d}{dt} C_1^0(t) \vec{r}^{\,1}_{12} \\
&= \dot{C}_1^0(t) \vec{r}^{\,1}_{12} \\
&= [\vec{\omega}^{\,0}_{01} \times] C_1^0(t) \vec{r}^{\,1}_{12} \\
&= \vec{\omega}^{\,0}_{01} \times \vec{r}^{\,0}_{12}(t)
\end{aligned}
$$

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Linear Acceleration

Linear acceleration:

$$
\begin{aligned}
\ddot{\vec{r}}^{\,0}_{02} &= \frac{d}{dt} \left(\vec{\omega}^{\,0}_{01} \times \left(C_1^0(t) \vec{r}^{\,1}_{12} \right) \right) \\
&= \dot{\vec{\omega}}^{\,0}_{01} \times \left(C_1^0(t) \vec{r}^{\,1}_{12} \right) + \vec{\omega}^{\,0}_{01} \times \left(\dot{C}_1^0(t) \vec{r}^{\,1}_{12} \right) \\
&= \underbrace{\boxed{\dot{\vec{\omega}}^{\,0}_{01} \times \vec{r}^{\,0}_{12}(t)}}_{\text{Transverse accel}} + \underbrace{\boxed{\vec{\omega}^{\,0}_{01} \times \left(\vec{\omega}^{\,0}_{01} \times \vec{r}^{\,0}_{12}(t) \right)}}_{\text{Centripetal accel } (\omega^2 r)}
\end{aligned}
$$

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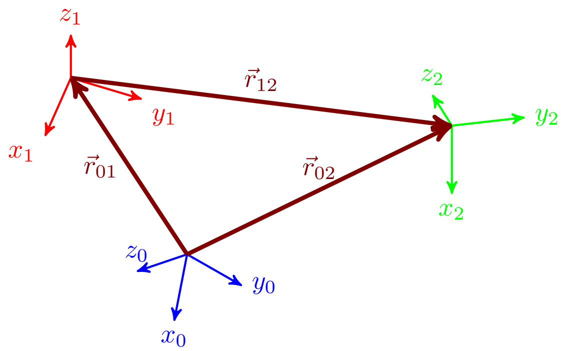
Linear Position

We can get back to where we started ... motion (translation and rotation) between frames and their derivatives.



Translation (position) between frames $\{0\}$ and $\{1\}$: .24

$$
\begin{aligned}
\vec{r}^{\,0}_{02} &= \vec{r}^{\,0}_{01} + \vec{r}^{\,0}_{12} \\
&= \vec{r}^{\,0}_{01} + C^0_1 \vec{r}^{\,1}_{12}
\end{aligned}
$$

Linear Velocity

Linear velocity:

$$
\begin{aligned}
\dot{\vec{r}}^{\,0}_{02}(t) &= \frac{d}{dt}\left(\vec{r}^{\,0}_{01} + C^0_1 \vec{r}^{\,1}_{12}\right) \\
&= \dot{\vec{r}}^{\,0}_{01} + \dot{C}^0_1 \vec{r}^{\,1}_{12} + C^0_1 \dot{\vec{r}}^{\,1}_{12} \\
&= \dot{\vec{r}}^{\,0}_{01} + \Omega^0_{01} C^0_1 \vec{r}^{\,1}_{12} + C^0_1 \dot{\vec{r}}^{\,1}_{12} \\
&= \dot{\vec{r}}^{\,0}_{01} + [\vec{\omega}^{\,0}_{01} \times] C^0_1 \vec{r}^{\,1}_{12} + C^0_1 \dot{\vec{r}}^{\,1}_{12} \\
&= \dot{\vec{r}}^{\,0}_{01} + \vec{\omega}^{\,0}_{01} \times (C^0_1 \vec{r}^{\,1}_{12}) + C^0_1 \dot{\vec{r}}^{\,1}_{12}
\end{aligned}
$$

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Linear Acceleration

Linear acceleration:

$$
\begin{aligned}
\ddot{\vec{r}}^{\,0}_{02} &= \frac{d}{dt}\left(\dot{\vec{r}}^{\,0}_{01} + \vec{\omega}^{\,0}_{01} \times (C^0_1 \vec{r}^{\,1}_{12}) + C^0_1 \dot{\vec{r}}^{\,1}_{12}\right) \\
&= \ddot{\vec{r}}^{\,0}_{01} + \dot{\vec{\omega}}^{\,0}_{01} \times (C^0_1 \vec{r}^{\,1}_{12}) + \vec{\omega}^{\,0}_{01} \times \left(\dot{C}^0_1 \vec{r}^{\,1}_{12}\right) + \vec{\omega}^{\,0}_{01} \times \left(C^0_1 \dot{\vec{r}}^{\,1}_{12}\right) + \dot{C}^0_1 \dot{\vec{r}}^{\,1}_{12} + C^0_1 \ddot{\vec{r}}^{\,1}_{12}
\end{aligned}
$$

$$
= \boxed{\ddot{\vec{r}}^{\,0}_{01}} + \boxed{\dot{\vec{\omega}}^{\,0}_{01} \times \vec{r}^{\,0}_{12}(t)} + \boxed{\vec{\omega}^{\,0}_{01} \times \left(\vec{\omega}^{\,0}_{01} \times \vec{r}^{\,0}_{12}(t)\right)} + \boxed{2\vec{\omega}^{\,0}_{01} \times \left(C^0_1 \dot{\vec{r}}^{\,1}_{12}\right)} + \boxed{C^0_1 \ddot{\vec{r}}^{\,1}_{12}}
$$

accel of $\{1\}$'s origin from $\{0\}$ in $\{0\}$

Centripetal accel ($\omega^2 r$)

Coriolis accel ($2\omega \times v$) .26

Transverse accel

accel of $\{2\}$'s origin from $\{1\}$ in $\{0\}$

The End

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