EE 565: Position, Navigation and Timing

Navigation Mathematics: Angular and Linear Velocity

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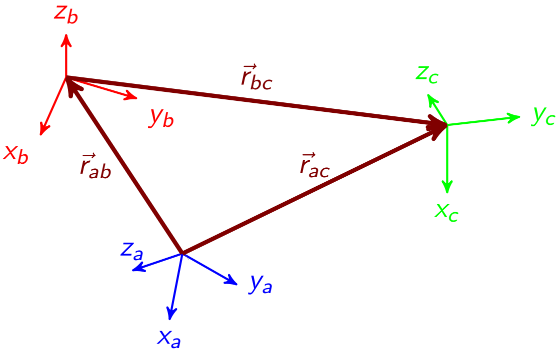
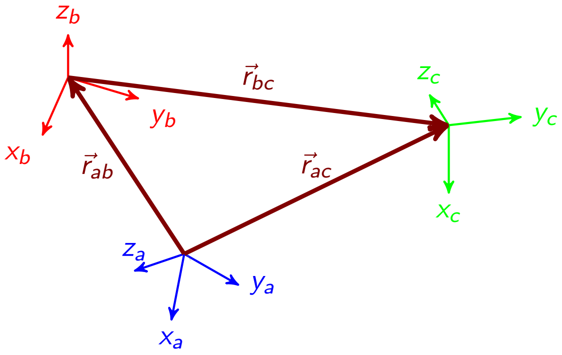


- translation between frames $\{a\}$ and $\{c\}$:

$$\vec{r}_{ac} = \vec{r}_{ab} + \vec{r}_{bc}$$



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$$\vec{r}_{ac} = \vec{r}_{ab} + \vec{r}_{bc}$$

- written *wrt*/frame $\{a\}$

$$\begin{aligned} \vec{r}^{\,a}_{ac} &= \vec{r}^{\,a}_{ab} + \vec{r}^{\,a}_{bc} \\ &= \vec{r}^{\,a}_{ab} + C^a_b \vec{r}^{\,b}_{bc} \end{aligned}$$

- Given relationship for translation between moving (rotating and translating) frames

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- Why is $\dot{C}^a_b \neq 0$ in general? Recoordinatization of $\vec{r}^{\,b}_{bc}$ is time-dependent.
- $\dot{C}^a_b$ is directly related to angular velocity between frames $\{a\}$ and $\{b\}$.

Given a rotation matrix C, one of its properties is

$$[C_b^a]^T C_b^a = C_b^a [C_b^a]^T = \mathcal{I}$$

Taking the time-derivative of the "right-inverse" property

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First approach to $\frac{d}{dt}C$ and angular velocity

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$$\Rightarrow \Omega_{ab}^a + [\Omega_{ab}^a]^T = 0$$

$$\Rightarrow \Omega_{ab}^a \text{ is skew-symmetric!}$$

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Define this skew-symmetric matrix Ω_{ab}^{a}

$$\Omega_{ab}^{a} = [\vec{\omega}\,_{ab}^{a}\times] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

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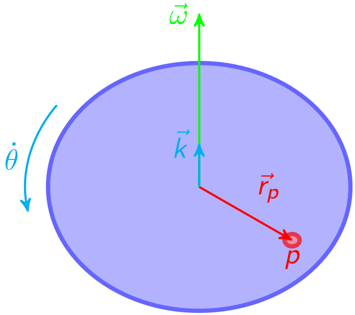
Note $\Omega^a_{ab} = \dot{C}^a_b [C^a_b]^T$

$$\Rightarrow \dot{C}^a_b = \Omega^a_{ab} C^a_b$$

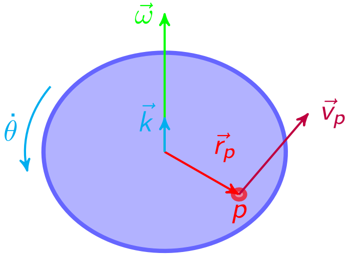
is a means of finding derivative of rotation matrix provided we can further understand Ω^a_{ab}.

Now for some insight into physical meaning of Ω^a_{ab}.

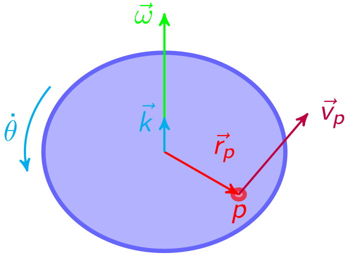
- Consider a point p on a rigid body rotating with angular velocity $\vec{\omega} = [\omega_x,\ \omega_y,\ \omega_z]^T = \dot{\theta}\vec{k} = \dot{\theta}[k_x,\ k_y,\ k_z]^T$ with $\vec{k}$ a unit vector.



First approach to $\frac{d}{dt}C$ and angular velocity

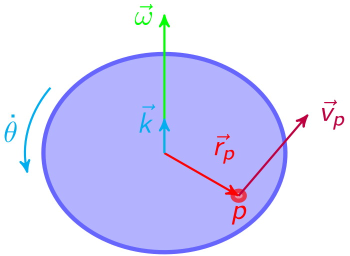


From mechanics, linear velocity $\vec{v}_p$ of point is



From mechanics, linear velocity $\vec{v}_p$ of point is

$$\vec{v}_p = \vec{\omega} \times \vec{r}_p = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \omega_y r_z - \omega_z r_y \\ \omega_z r_x - \omega_x r_z \\ \omega_x r_y - \omega_y r_x \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}_{?} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$



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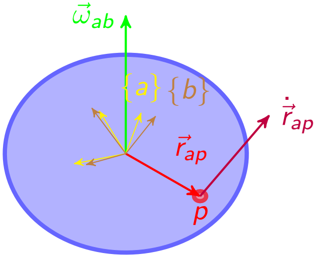
$\Rightarrow \Omega$ represents angular velocity and performs cross product

Now let's add fixed frame $\{a\}$ and rotating frame $\{b\}$ attached to moving body such that there is angular velocity $\vec{\omega}_{ab}$ between them.



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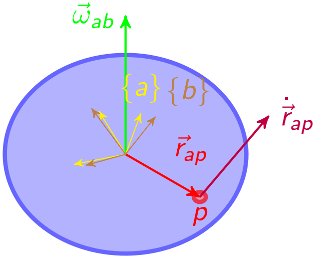

Start with position

$$\vec{r}^{\,a}_{ap} = \underbrace{\vec{r}^{\,a}_{ab}}_{0} + C^a_b \vec{r}^{\,b}_{bp}$$

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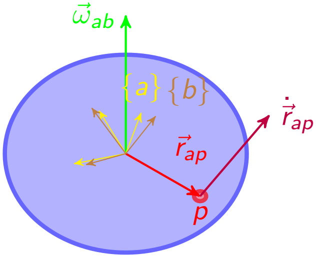

and take derivative *wrt* time

$$\begin{aligned}\dot{\vec{r}}^{\,a}_{ap} &= \underbrace{\dot{C}^a_b}_{\Omega^a_{ab} C^a_b} \vec{r}^{\,b}_{bp} + \underbrace{C^a_b \dot{\vec{r}}^{\,b}_{bp}}_{0} \\ &= \Omega^a_{ab} C^a_b \vec{r}^{\,b}_{bp} \\ &= \Omega^a_{ab} \vec{r}^{\,a}_{bp} = [\vec{\omega}^{\,a}_{ab} \times] \vec{r}^{\,a}_{bp}\end{aligned}$$

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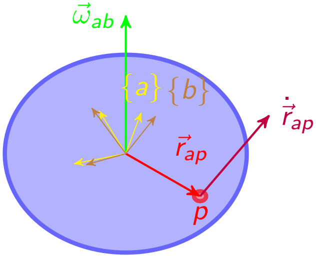

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\dot{\vec{r}}^{\,a}_{ap} &= \underbrace{\dot{C}^a_b}_{\Omega^a_{ab}C^a_b}\vec{r}^{\,b}_{bp} + \underbrace{C^a_b\dot{\vec{r}}^{\,b}_{bp}}_{0} \\
&= \Omega^a_{ab}C^a_b\vec{r}^{\,b}_{bp} \\
&= \Omega^a_{ab}\vec{r}^{\,a}_{bp} = [\vec{\omega}^{\,a}_{ab}\times]\vec{r}^{\,a}_{bp}
\end{aligned}$$

Start with position

$$\vec{r}^{\,a}_{ap} = \underbrace{\vec{r}^{\,a}_{ab}}_{0} + C^a_b\vec{r}^{\,b}_{bp}$$

from which it is observed (compare to $\vec{v}_p = \vec{\omega} \times \vec{r}_p$) that Ω^a_{ab} represents cross product with angular velocity $\vec{\omega}^{\,a}_{ab}$.

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- Another approach to developing derivative of rotation matrix and angular velocity is based upon angle-axis representation of orientation and rotation matrix as exponential.
- This approach is included in notes.

$$\begin{aligned} C\Omega C^T\vec{b} &= C\left[\vec{\omega}\times\left(C^T\vec{b}\right)\right] \\ &= C\vec{\omega}\times\left(CC^T\vec{b}\right) \\ &= C\vec{\omega}\times\vec{b} \\ &= [C\vec{\omega}\times]\vec{b} \end{aligned}$$

Therefore (from above),

$$C\Omega C^T = C[\vec{\omega}\times]C^T = [C\vec{\omega}\times]$$

and (via distributive property)

$$C[\vec{\omega}\times] = [C\vec{\omega}\times]C$$

noting both $\vec{\omega}$ and vector with which cross-product will be taken are assumed to be in the same coordinate frame and thus both need to be recoordinatized.

$$
\begin{aligned}
\dot{C}_b^a &= \Omega_{ab}^a C_b^a \\
&= [\vec{\omega}\,{}_{ab}^{a}\times] C_b^a \\
&= [C_b^a \vec{\omega}\,{}_{ab}^{b}\times] C_b^a \\
&= C_b^a [\vec{\omega}\,{}_{ab}^{b}\times] \\
&= C_b^a \Omega_{ab}^b
\end{aligned}
$$

$$\Rightarrow \dot{C}_b^a = \Omega_{ab}^a C_b^a = C_b^a \Omega_{ab}^b$$

Angular velocity can be

- described as a vector
  - the angular velocity of the b–frame *wrt* the a–frame resolved in the c–frame, $\vec{\omega}^{\,c}_{ab}$
  - $\vec{\omega}_{ab} = -\vec{\omega}_{ba}$

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- described as a skew-symmetric matrix $\Omega^c_{ab} = [\vec{\omega}^{\,c}_{ab}\times]$
  - the skew-symmetric matrix is equivalent to the vector cross product when pre-multiplying another vector

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- described as a skew-symmetric matrix $\Omega^c_{ab} = [\vec{\omega}^{\,c}_{ab}\times]$
  - the skew-symmetric matrix is equivalent to the vector cross product when pre-multiplying another vector
- related to the derivative of the rotation matrix

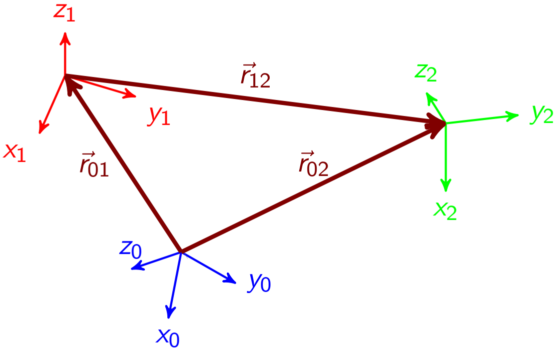
$$\dot{C}^a_b = \Omega^a_{ab} C^a_b = C^a_b \Omega^b_{ab}$$
$$\dot{C}^a_b = -\Omega^a_{ba} C^a_b = -C^a_b \Omega^b_{ba}$$

Consider the derivative of the composition of rotations $C_2^0 = C_1^0 C_2^1$.

$$
\begin{aligned}
\tfrac{d}{dt} C_2^0 &= \frac{d}{dt} C_1^0 C_2^1 \\
\Rightarrow \quad \dot{C}_2^0 &= \dot{C}_1^0 C_2^1 + C_1^0 \dot{C}_2^1 \\
\Rightarrow \quad \Omega_{02}^0 C_2^0 &= \Omega_{01}^0 C_1^0 C_2^1 + C_1^0 C_2^1 \Omega_{12}^2 \\
\Rightarrow \quad \Omega_{02}^0 &= \Omega_{01}^0 C_2^0 \left[C_2^0\right]^T + C_2^0 \Omega_{12}^2 \left[C_2^0\right]^T \\
\Rightarrow \quad [\vec{\omega}\,_{02}^{\,0} \times] &= [\vec{\omega}\,_{01}^{\,0} \times] + [C_2^0 \vec{\omega}\,_{12}^{\,2} \times] \\
\Rightarrow \quad \vec{\omega}\,_{02}^{\,0} &= \vec{\omega}\,_{01}^{\,0} + \vec{\omega}\,_{12}^{\,0}
\end{aligned}
$$

$\Rightarrow$ angular velocities (as vectors) add so long as resolved common coordinate system

We can get back to where we started ... motion (translation and rotation) between frames and their derivatives.



Translation (position) between frames $\{0\}$ and $\{1\}$:

$$\begin{aligned} \vec{r}^{\,0}_{02} &= \vec{r}^{\,0}_{01} + \vec{r}^{\,0}_{12} \\ &= \vec{r}^{\,0}_{01} + C^0_1 \vec{r}^{\,1}_{12} \end{aligned}$$

Linear velocity:

$$\begin{aligned}
\dot{\vec{r}}_{02}^{\,0}(t) &= \frac{d}{dt}\left(\vec{r}_{01}^{\,0} + C_1^0 \vec{r}_{12}^{\,1}\right) \\
&= \dot{\vec{r}}_{01}^{\,0} + \dot{C}_1^0 \vec{r}_{12}^{\,1} + C_1^0 \dot{\vec{r}}_{12}^{\,1} \\
&= \dot{\vec{r}}_{01}^{\,0} + \Omega_{01}^0 C_1^0 \vec{r}_{12}^{\,1} + C_1^0 \dot{\vec{r}}_{12}^{\,1} \\
&= \dot{\vec{r}}_{01}^{\,0} + [\vec{\omega}_{01}^{\,0} \times] C_1^0 \vec{r}_{12}^{\,1} + C_1^0 \dot{\vec{r}}_{12}^{\,1} \\
&= \dot{\vec{r}}_{01}^{\,0} + \vec{\omega}_{01}^{\,0} \times \left(C_1^0 \vec{r}_{12}^{\,1}\right) + C_1^0 \dot{\vec{r}}_{12}^{\,1}
\end{aligned}$$

Linear acceleration:

$$
\begin{aligned}
\ddot{\vec{r}}^{\mathbf{0}}_{\mathbf{02}} &= \frac{d}{dt}\left(\dot{\vec{r}}^{\mathbf{0}}_{\mathbf{01}} + \vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \left(C^{\mathbf{0}}_{\mathbf{1}}\vec{r}^{\mathbf{1}}_{\mathbf{12}}\right) + C^{\mathbf{0}}_{\mathbf{1}}\dot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}}\right) \\
&= \ddot{\vec{r}}^{\mathbf{0}}_{\mathbf{01}} + \dot{\vec{\omega}}^{\mathbf{0}}_{\mathbf{01}} \times \left(C^{\mathbf{0}}_{\mathbf{1}}\vec{r}^{\mathbf{1}}_{\mathbf{12}}\right) + \vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \left(\dot{C}^{\mathbf{0}}_{\mathbf{1}}\vec{r}^{\mathbf{1}}_{\mathbf{12}}\right) + \vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \left(C^{\mathbf{0}}_{\mathbf{1}}\dot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}}\right) + \dot{C}^{\mathbf{0}}_{\mathbf{1}}\dot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}} + C^{\mathbf{0}}_{\mathbf{1}}\ddot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}} \\
&= \ddot{\vec{r}}^{\mathbf{0}}_{\mathbf{01}} + \dot{\vec{\omega}}^{\mathbf{0}}_{\mathbf{01}} \times \vec{r}^{\mathbf{0}}_{\mathbf{12}}(t) + \vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \left(\vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \vec{r}^{\mathbf{0}}_{\mathbf{12}}(t)\right) + 2\vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \left(C^{\mathbf{0}}_{\mathbf{1}}\dot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}}\right) + C^{\mathbf{0}}_{\mathbf{1}}\ddot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}}
\end{aligned}
$$

accel of {1}'s origin from {0} in {0}

Transverse accel

Centripetal accel ($\omega^2 r$)

Coriolis accel ($2\omega \times v$)

accel of {2}'s origin from {1} in {0}

The End