EE 565: Position, Navigation and Timing

Navigation Mathematics: Kinematics (Angular Velocity: Quaternion Representation)

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February 11, 2020

Lecture Topics

1. Review
   - Useful Quaternion Properties
   - Angular Velocity
2. Angular Velocity Using Quaternions

- DCM (9-elements), e.g., $C_1^2 C_0^1$
- Quaternion (4-elements), e.g., $\bar{q}\,_1^2 \otimes \bar{q}\,_0^1$
- Recoordinatizing a vector using DCM, e.g., $\vec{r}^{\,2} = C_1^2 \vec{r}^{\,1}$
- Recoordinatizing a vector using quaternion, e.g., $\check{r}^2 = \bar{q}\,_1^2 \otimes \check{r}^1 \otimes (\bar{q}\,_1^2)^{-1}$, where $\check{r} = [0 \quad \vec{r}]^T$

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- Recoordinatizing a vector using DCM, e.g., $\vec{r}^{\,2} = C_1^2 \vec{r}^{\,1}$
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How many additions and multiplications does each of the above computations require?

Quaternion Multiply

- Quaternion multiply

$$\bar{r} = \bar{q} \otimes \bar{p} = [\bar{q}\otimes]\bar{p} = \begin{bmatrix} q_s p_s - \vec{q} \cdot \vec{p} \\ q_s\vec{p} + p_s\vec{q} + \vec{q} \times \vec{p} \end{bmatrix}$$

where

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

- Quaternion multiply (corresponds to reverse order DCM)

$$\bar{r} = \bar{q} \circledast \bar{p} = [\bar{q}\circledast]\bar{p} = \begin{bmatrix} q_s p_s - \vec{q} \cdot \vec{p} \\ q_s \vec{p} + p_s \vec{q} - \vec{q} \times \vec{p} \end{bmatrix}$$

- $$\bar{q} \otimes \bar{p} = \bar{p} \circledast \bar{q}$$

  where

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

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$$\dot{C}_b^a = \Omega_{ab}^a C_b^a \tag{1}$$

where

$$\Omega_{ab}^a = [\vec{\omega}\,{}_{ab}^a \times] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Angular Velocity using Angle Axis

- Since the relative and fixed axis rotations must be performed in a particular order, their derivatives are somewhat challenging
- The angle-axis format, however, is readily differentiable as we can encode the 3 parameters by

$$\vec{K} \equiv \vec{k}(t)\theta(t) = \begin{bmatrix} K_1(t) \\ K_2(t) \\ K_3(t) \end{bmatrix} \tag{2}$$

  where $\theta = \|\vec{K}\|$
- Hence,

$$\frac{d}{dt}\vec{K}(t) = \begin{bmatrix} \dot{K}_1(t) \\ \dot{K}_2(t) \\ \dot{K}_3(t) \end{bmatrix}$$

- For a sufficiently "small" time interval we can often consider the axis of rotation to be $\approx$ constant (i.e.,$\vec{K}(t) = \vec{k}$)

$$\frac{d}{dt}\vec{K}(t) = \frac{d}{dt}\left(\vec{k}\theta(t)\right)$$
$$= \vec{k}\dot{\theta}(t)$$

- This is referred to as the angular velocity ($\vec{\omega}(t)$) or the so called "body reference" angular velocity

Angular Velocity

$$\vec{\omega}(t) \equiv \vec{k}\dot{\theta}(t) \tag{3}$$

Motivation for Using Quaternions

- Require minimal storage
- Offer computational advantages over other methods
- Lack of singularities

Angular Velocity Using Quaternions

- Recalling that

$$[\bar{q}\,{}^{a}_{b}(t)\otimes] = e^{\frac{1}{2}[\breve{k}^{a}_{ab}\otimes]\theta(t)} = \cos(\theta/2)\mathcal{I} + \frac{1}{2}[\breve{k}^{a}_{ab}\otimes]\frac{\sin(\theta/2)}{\theta/2}$$

where

$$\breve{k} = \begin{bmatrix} 0 \\ \vec{k} \end{bmatrix}$$

- Hence,

$$\begin{aligned}
\frac{d}{dt}[\bar{q}\,{}^{a}_{b}(t)\otimes] &= \frac{d}{dt}e^{\frac{1}{2}[\breve{k}^{a}_{ab}\otimes]\theta(t)} = \frac{\partial e^{\frac{1}{2}[\breve{k}^{a}_{ab}\otimes]\theta(t)}}{\partial\theta}\frac{d\theta}{dt} \\
&= \frac{1}{2}[\breve{k}^{a}_{ab}\otimes]e^{\frac{1}{2}[\breve{k}^{a}_{ab}\otimes]\theta(t)}\dot{\theta}(t) \\
&= \frac{1}{2}\left([\breve{k}^{a}_{ab}\otimes]\dot{\theta}(t)\right)[\bar{q}\,{}^{a}_{b}(t)\otimes]
\end{aligned}$$

Angular Velocity Using Quaternions

- let $W^a_{ab} = \left([\breve{k}^a_{ab}\otimes]\dot{\theta}(t)\right) = [\breve{\omega}^a_{ab}\otimes]$
- therefore,

$$W^a_{ab} = \begin{bmatrix} 0 & -\omega^a_{ab,x} & -\omega^a_{ab,y} & -\omega^a_{ab,z} \\ \omega^a_{ab,x} & 0 & -\omega^a_{ab,z} & \omega^a_{ab,y} \\ \omega^a_{ab,y} & \omega^a_{ab,z} & 0 & -\omega^a_{ab,x} \\ \omega^a_{ab,z} & -\omega^a_{ab,y} & \omega^a_{ab,x} & 0 \end{bmatrix}$$

- and consequently,

$$\dot{\bar{q}}^a_b(t) = \frac{1}{2}[\breve{\omega}^a_{ab}\otimes]\bar{q}^{\,a}_{\,b}(t)$$

- Now,

$$\begin{aligned}\dot{\bar{q}}^a_b(t) &= \frac{1}{2}[\breve{\omega}^a_{ab}\otimes]\bar{q}^{\,a}_{\,b}(t) = \frac{1}{2}\breve{\omega}^a_{ab}\otimes\bar{q}^{\,a}_{\,b}(t)\\ &= \frac{1}{2}\bar{q}^{\,a}_{\,b}(t)\otimes\breve{\omega}^b_{ab}\otimes(\bar{q}^{\,a}_{\,b}(t))^{-1}\otimes\bar{q}^{\,a}_{\,b}(t)\\ &= \frac{1}{2}[\bar{q}^{\,a}_{\,b}(t)\otimes]\breve{\omega}^b_{ab}\\ &= \frac{1}{2}[\breve{\omega}^b_{ab}\circledast]\bar{q}^{\,a}_{\,b}(t)\end{aligned}$$

$$\breve{\omega}^a_{ab} = \bar{q}^{\,a}_{\,b}(t)\otimes\breve{\omega}^b_{ab}\otimes(\bar{q}^{\,a}_{\,b}(t))^{-1}$$

$$(\bar{q}^{\,a}_{\,b}(t))^{-1}\otimes\bar{q}^{\,a}_{\,b}(t) = 1$$

- and consequently,

$$\dot{\bar{q}}^a_b(t) = \frac{1}{2}[\breve{\omega}^a_{ab}\otimes]\bar{q}^{\,a}_{\,b}(t) = \frac{1}{2}[\bar{q}^{\,a}_{\,b}(t)\otimes]\breve{\omega}^b_{ab} = \frac{1}{2}[\breve{\omega}^b_{ab}\circledast]\bar{q}^{\,a}_{\,b}(t) \tag{4}$$