Lecture

Navigation Equations: ECI Mechanization

EE 565: Position, Navigation and Timing

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ECI Mechanization

- Determine the position, velocity and attitude of the **body** frame *wrt* the **inertial** frame
  - **Position** — Vector from the origin of the inertial frame to the origin of the body frame resolved in the inertial frame: $\vec{r}^{\,i}_{ib}$
  - **Velocity** — Velocity of the body frame *wrt* the inertial frame resolved in the inertial frame: $\vec{v}^{\,i}_{ib}$
  - **Attitude** — Orientation of the body frame *wrt* the inertial frame: C^i_b or $\bar{q}^{\,i}_b$
- The inputs are $\vec{\omega}^{\,b}_{ib}$ and $\vec{f}^{\,b}_{ib}$

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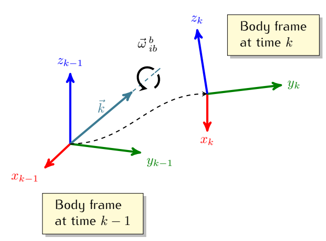
Attitude — Method A

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$

$$\begin{aligned}\dot{C}^i_b &= C^i_b \Omega^b_{ib} \\ &= \lim_{\Delta t \to 0} \left(\frac{C^i_b(k) - C^i_b(k-1)}{\Delta t} \right) = C^i_b(k-1)\Omega^b_{ib}\end{aligned}$$

$$C^i_b(+) - C^i_b(-) \approx C^i_b(-)\Omega^b_{ib}\Delta t$$

$$C^i_b(+) \approx C^i_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right)$$



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Attitude — Method B

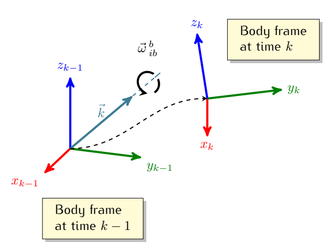
- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$

$$\boxed{\vec{\omega}^{\,b}_{ib}\Delta t = \vec{k}\Delta\theta}$$

$$C^i_{b(k)} = C^i_{b(k-1)} C^{b(k-1)}_{b(k)}$$

$$\begin{aligned}C^{b(k-1)}_{b(k)} &= e^{\Omega^b_{ib}\Delta t} = e^{\mathfrak{K}\Delta\theta} \\ &= \mathcal{I} + \mathfrak{K}\Delta\theta + \frac{\mathfrak{K}^2\Delta\theta^2}{2!} + \frac{\mathfrak{K}^3\Delta\theta^3}{3!} + \dots \\ &= \mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + \left[1 - \cos(\Delta\theta)\right]\mathfrak{K}^2\end{aligned}$$

$$\begin{aligned}C^i_b(+) &= C^i_b(-) C^{b(k-1)}_{b(k)} \\ &\approx C^i_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right)\end{aligned}$$



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$$\boxed{\mathfrak{K} = [\vec{k}\times]}$$

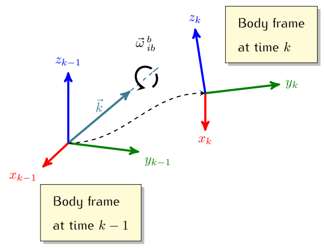
Attitude — Method C

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$ $\boxed{\vec{\omega}^{\,b}_{ib}\Delta t = \vec{k}\Delta\theta}$

$$\bar{q}^{\,i}_{\,b(k)} = \bar{q}^{\,i}_{\,b(k-1)} \otimes \bar{q}^{\,b(k-1)}_{\,b(k)}$$

$$\bar{q}^{\,b(k-1)}_{\,b(k)} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$

$$\bar{q}^{\,i}_{\,b}(+) = \bar{q}^{\,i}_{\,b}(-) \otimes \bar{q}^{\,b(k-1)}_{\,b(k)}$$



Need to periodically renormalize $\bar{q}$

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Attitude Update— Summary

- High fidelity

$$C^i_b(+) = C^i_b(-)\left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right] \tag{1}$$

or

$$\bar{q}^{\,i}_{\,b}(+) = \bar{q}^{\,i}_{\,b}(-) \otimes \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \tag{2}$$

- Low fidelity

$$C^i_b(+) \approx C^i_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) \tag{3}$$

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Steps 2–4

2. Specific force transformation
   - Simply coordinatize the specific force

$$\vec{f}^{\,i}_{\,ib} = C^i_b(+)\vec{f}^{\,b}_{\,ib} \tag{4}$$

3. Velocity update
   - Assuming that we are in space (i.e., no centrifugal component)

$$\vec{a}^{\,i}_{\,ib} = \vec{f}^{\,i}_{\,ib} + \vec{\gamma}^{\,i}_{\,ib} \tag{5}$$

   - Thus, by simple numerical integration

$$\vec{v}^{\,i}_{\,ib}(+) = \vec{v}^{\,i}_{\,ib}(-) + \vec{a}^{\,i}_{\,ib}\Delta t \tag{6}$$

4. Position update
   - by simple numerical integration

$$\vec{r}^{\,i}_{\,ib}(+) = \vec{r}^{\,i}_{\,ib}(-) + \vec{v}^{\,i}_{\,ib}(-)\Delta t + \vec{a}^{\,i}_{\,ib}\frac{\Delta t^2}{2} \tag{7}$$

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ECI Mechanization Summary

$$C_b^i(+) = C_b^i(-)\left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right]$$

or

$$C_b^i(+) \approx C_b^i(-)\left(\mathcal{I} + \Omega_{ib}^b \Delta t\right)$$

or

$$\bar{q}\,_b^i(+) = \bar{q}\,_b^i(-) \otimes \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$

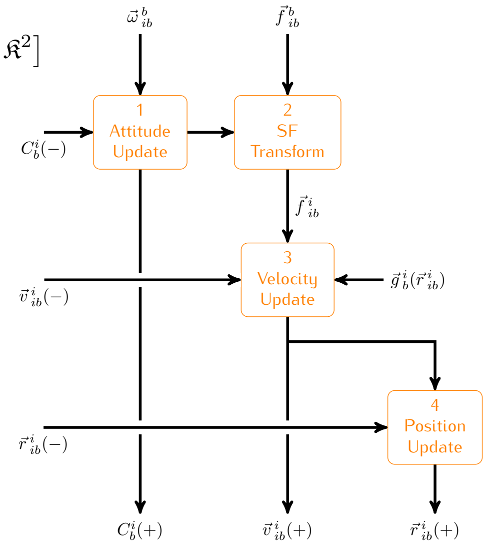
and

$$\vec{f}\,_{ib}^{\,i} = C_b^i(+)\vec{f}\,_{ib}^{\,b}$$

$$\vec{a}\,_{ib}^{\,i} = \vec{f}\,_{ib}^{\,i} + \vec{\gamma}\,_{ib}^{\,i}$$

$$\vec{v}\,_{ib}^{\,i}(+) = \vec{v}\,_{ib}^{\,i}(-) + \vec{a}\,_{ib}^{\,i}\Delta t$$

$$\vec{r}\,_{ib}^{\,i}(+) = \vec{r}\,_{ib}^{\,i}(-) + \vec{v}\,_{ib}^{\,i}(-)\Delta t + \vec{a}\,_{ib}^{\,i}\frac{\Delta t^2}{2}$$



What is the importance of Δt? .8

ECI Mechanization — Continuous Case

- In continuous time notation
  - Attitude: $\dot{C}_b^i = C_b^i\Omega_{ib}^b$ or $\dot{\bar{q}}_b^i = \frac{1}{2}[\breve{\omega}_{ib}^b \circledast]\bar{q}\,_b^i(t)$
  - Velocity: $\dot{\vec{v}}\,_{ib}^{\,i} = C_b^i\vec{f}\,_{ib}^{\,b} + \vec{\gamma}\,_{ib}^{\,i}$
  - Position: $\dot{\vec{r}}\,_{ib}^{\,i} = \vec{v}\,_{ib}^{\,i}$
- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}\,_{ib}^{\,i} \\ \dot{\vec{v}}\,_{ib}^{\,i} \\ \dot{C}_b^i \end{bmatrix} = \begin{bmatrix} \vec{v}\,_{ib}^{\,i} \\ C_b^i\vec{f}\,_{ib}^{\,b} + \vec{\gamma}\,_{ib}^{\,i} \\ C_b^i\Omega_{ib}^b \end{bmatrix} \tag{8}$$

or

$$\begin{bmatrix} \dot{\vec{r}}\,_{ib}^{\,i} \\ \dot{\vec{v}}\,_{ib}^{\,i} \\ \dot{\bar{q}}_b^i \end{bmatrix} = \begin{bmatrix} \vec{v}\,_{ib}^{\,i} \\ C_b^i\vec{f}\,_{ib}^{\,b} + \vec{\gamma}\,_{ib}^{\,i} \\ \frac{1}{2}[\breve{\omega}_{ib}^b \circledast]\bar{q}\,_b^i(t) \end{bmatrix} \tag{9}$$

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Appendix

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

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