Lecture

Power Spectral Density Estimation

EE 565: Position, Navigation, and Timing

Lecture Notes Update on March 11, 2020

Aly El-Osery and Kevin Wedeward, Electrical Engineering Dept., New Mexico Tech
In collaboration with
Stephen Bruder, Electrical & Computer Engineering, Embry-Riddle Aeronautical University

.1

Motivation

Sensors suffer from noise effects that can not be removed through calibration, consquently, we need to

- understand the nature of the noise
- be able to extract parameters from actual data
- develop models to mimic noise in simulation to provide performance capabilities

.2

Purpose

Estimate the distribution of power in a signal. Unfortunately, truth and what is practical cause a problem.

Truth

- Infinitely long.
- Continuous in time and value.
- Provides true distribution of power.

Practice

- Finite length.
- Discrete in time and value.
- Only approximation of distribution of power.

Let's make it more interesting
The signal is stochastic in nature.

.3

1 Review Material

1.1 Signal Classification

Energy and Power

Assume the voltage across a resistor R is $e(t)$ and is producing a current $i(t)$. The instantaneous power per ohm is $p(t) = e(t)i(t)/R = i^2(t)$.

Total Energy

$$E = \lim_{T\to\infty} \int_{-T}^{T} i^2(t)dt \tag{1}$$

Average Power

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)dt \tag{2}$$

.4

Arbitrary signal $x(t)$

Total Normalized Energy

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{3}$$

Normalized Power

$$P \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{4}$$

- $x(t)$ is an *energy signal* iff $0 < E < \infty$, so that $P = 0$.
- $x(t)$ is a *power signal* iff $0 < P < \infty$, so that $E = \infty$.

.5

1.2 Time Averages

Correlation

For Energy Signals

$$\phi(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \tag{5}$$

Provides a measure of similarity or coherence between a signal and a delayed version of itself. Note that $\phi(0) = E$

For Power Signals

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt \tag{6}$$

For Periodic Signals

$$R(\tau) = \frac{1}{T_0} \int_{T_0} x(t)x(t+\tau)dt \tag{7}$$

.6

1.3 Frequency Domain

Energy Spectral Density

Rayleigh's Energy Theorem or Parseval's theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF \tag{8}$$

Energy Spectral Density

$$G(F) \triangleq |X(F)|^2 \tag{9}$$

with units of *volts*^{2}-*sec*^{2} or, if considered on a per-ohm basis, *watts-sec/Hz=joules/Hz*

.7

Power Spectral Density

$$P = \int_{-\infty}^{\infty} S(F)dF = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{10}$$

where we define $S(F)$ as the power spectral density with units of watts/Hz. Note that $R(0) = \int_{-\infty}^{\infty} S(F)dF$. .8

2 Random Signals and Noise

Basic Definitions

- Define an *experiment* with random *outcome*.
- Mapping of the outcome to a variable $\Rightarrow$ random variable.
- Mapping of the outcome to a function $\Rightarrow$ random function.

.9

2.1 Statistical Averages

Probability (Cumulative) Distribution Function (cdf)

$$F_X(x) = \texttt{probability that } X \leq x = P(X \leq x) \tag{11}$$

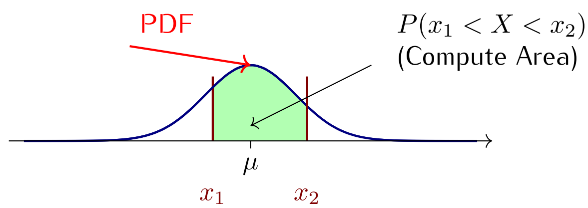
Describes the manner random variables take different values. .10

Probability Density Function (pdf)

$$f_X(x) = \frac{dF_X(x)}{dx} \tag{12}$$

and

$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx \tag{13}$$



.11

PDF of Discrete Random Variables

If the random variable X takes a set of discrete values x_i with probability p_i, the pdf of X is expressed in terms of Dirac delta functions, i.e.,

$$f_X(x) = \sum_i p_i \delta(x - x_i) \tag{14}$$

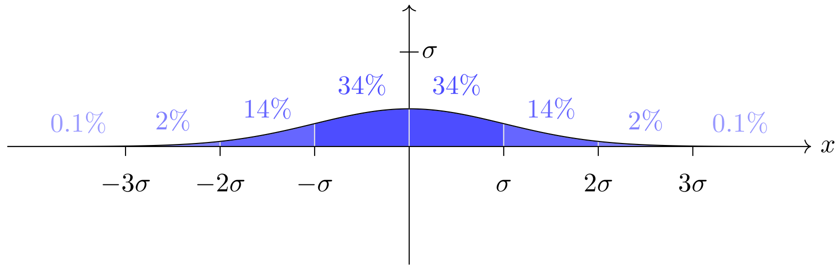
.12

Gaussian Distribution

$$f_X(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left[-\frac{x-\mu_x}{2\sigma_x^2}\right] \tag{15}$$

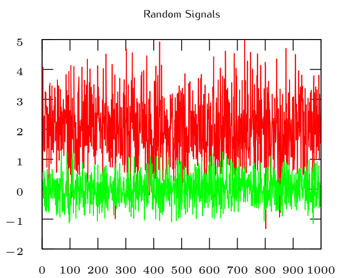
For example if $\sigma_x = \sigma$ and $\mu_x = 0$

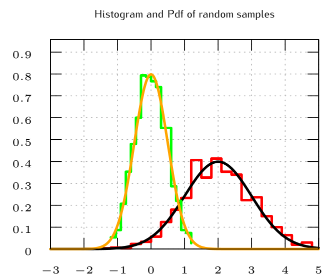
$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$



.13

PDF of White Noise





.14

Mean and Variance

Mean of a Discrete RV

$$\bar{X} = \mathbb{E}[X] = \sum_{j=1}^{M} x_j P_j \tag{16}$$

Mean of a Continuous RV

$$\bar{X} = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{17}$$

Variance of a RV

$$\sigma_X^2 \triangleq \mathbb{E}\left\{[X - \mathbb{E}(X)]^2\right\} = \mathbb{E}[X^2] - \mathbb{E}^2[X] \tag{18}$$

.15

Covariance and Autocorrelation

Given a two random variables X and Y.

Covariance

$$\mu_{XY} = \mathbb{E}\left\{[X - \bar{x}][Y - \bar{Y}]\right\} = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \tag{19}$$

Correlation Coefficient

$$\rho_{XY} = \frac{\mu_{XY}}{\sigma_X \sigma_Y} \tag{20}$$

Autocorrelation

$$\Gamma_X(\tau) = \mathbb{E}[X(t)X(t+\tau)] \tag{21}$$

.16

2.2 Stochastic Processes

Terminology

See Figure 1

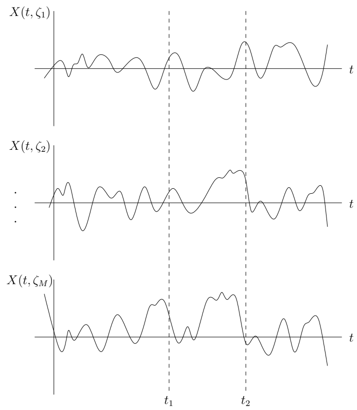


Figure 1: Sample functions of a random process

- $X(t, \zeta_i)$: sample function.
- The governing experiment: random or stochastic process.
- All sample functions: ensemble.
- $X(t_j, \zeta)$: random variable.

.17

Strict Sense Stationarity

If the joint pdfs depend only on the time difference regardless of the time origin, then the random process is known as *stationary*.

For stationary process means and variances are independent of time and the covariance depends only on the time difference. .18

Wide Sense Stationarity

If the joint pdfs depends on the time difference but the mean and variances are time-independent, then the random process is known as *wide-sense-stationary*.

.19

Ergodicity

If the time statistics equals ensemble statistics, then the random process is known as *ergodic*.

Any statistic calculated by averaging of all members of an ergodic ensemble at a fixed time can also be calculated by using a single representative waveform and averging over all time. .20

2.3 Correlation and Power Spectral Density

Power Spectral Density

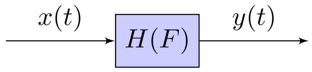
Given a sample function $X(t, \zeta_i)$ of a random process, we obtain the power spectral density by

$$S(F) \overset{\mathcal{F}}{\longleftrightarrow} \Gamma(\tau) \tag{22}$$

i.e., for a wide sense stationary signal, the power spectral density and autocorrelation are Fourier transform pairs. .21

2.4 Input-Output Relationship of Linear Systems

Input-Output Relationship of Linear Systems



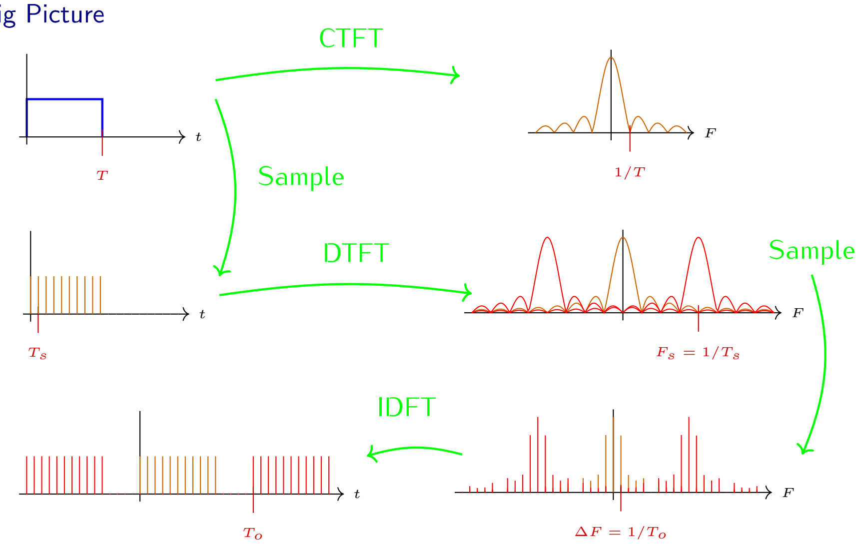
$$S_Y(F) = |H(F)|^2 S_X(F) \tag{23}$$

Noise Shaping
If $x(t)$ is white noise, we can design the filter $h(t)$ to “shape” the noise.

.22

3 Discrete Signals and Systems

Big Picture



.23

Sampling Remarks

- Must sample more than twice bandwidth to avoid aliasing.
- FFT represents a periodic version of the time domain signal → could have time domain aliasing.
- Number of points in FFT is the same as number of points in time domain signal.

.24

4 Power Spectral Density

Obtaining PSD for Discrete Signals

What we want is

$$\Gamma_X(\tau) = \mathbb{E}[X(t)X(t+\tau)] \xrightarrow{\mathcal{CTFT}} S_X(F)$$

For infinitely long signals.

What we can compute is

$$\gamma_X(m) = \mathbb{E}[X(n)X(n+m)] \xrightarrow{\mathcal{DFT}} P_X(f)$$

For finite length signals.

.25

What do we need in an estimate

As $N \to \infty$ and in the mean squared sense

Unbiased
Asymptotically the mean of the estimate approaches the true power.

Variance
Variance of the estimate approaches zero.

Resulting in a consistent estimate of the power spectrum. .26

Possible PSD Options

Periodogram
computed using $1/N$ times the magnitude squared of the FFT

$$\lim_{N\to\infty} \mathbb{E}[P_X(f)] = S_X(f)$$

$$\lim_{N\to\infty} var[P_X(f)] = S_X^2(f)$$

Welch Method
computed by segmenting the data (allowing overlaps), windowing the data in each segment then computing the average of the resultant priodogram

$$\mathbb{E}[P_X(f)] = \frac{1}{2\pi MU} S_X(f) \circledast W(f)$$

$$var[P_X(f)] \approx \frac{9}{8L} S_X^2(f)$$

.27

Welch Method

Assuming data length N, segment length M, Bartlett window, and 50% overlap

- FFT length $= M = 1.28/\Delta f = 1.28F_s/\Delta F$
- Resulting number of segments $= L = \frac{2N}{M}$
- Length of data collected in sec. $= \frac{1.28L}{2\Delta F}$

.28

pwelch Function

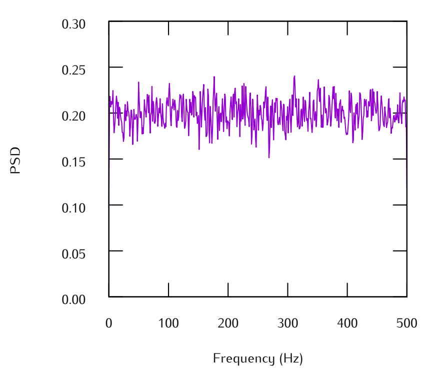
```
[Pxx,f] = pwelch(x,window,noverlap,...
                 nfft,fs,'range')
```

You can use [] in fields that you want the default to be used.

.29

pwelch Function - WGN signal

```
Fs = 1000;
x = sqrt(0.1*Fs)*randn(1,100000);
[Pxx,f] = pwelch(x,1024,[],[],Fs,'onesided');
```

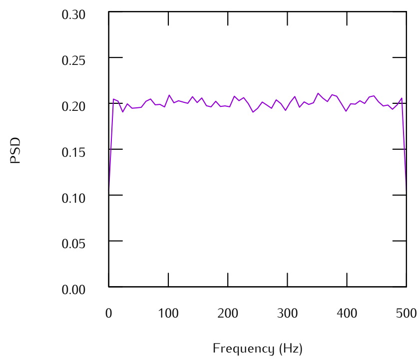


- Variance to high.

.30

pwelch Function - WGN signal

```
[Pxx,f] = pwelch(x,128,[],[],Fs,'onesided')
```

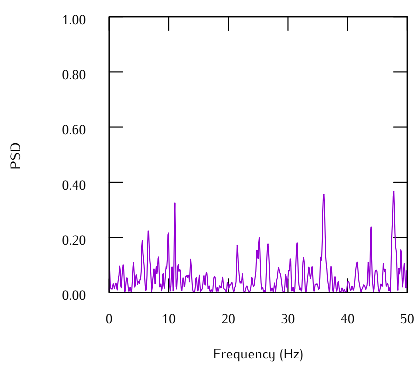


- Reduced window size.
- Variance is now smaller.

.31

pwelch Function - cos + WGN signal

```
Fs = 100;   t = 0:1/Fs:5;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,1024,[],[],Fs,'onesided');
```

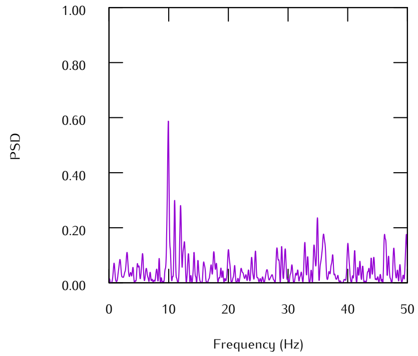


- Window larger than length of data.
- Frequency components can't be resolved.
- Variance high.

.32

pwelch Function - cos + WGN signal

```
Fs = 100;   t = 0:1/Fs:5;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,1024,[],4096,Fs,'onesided');
```

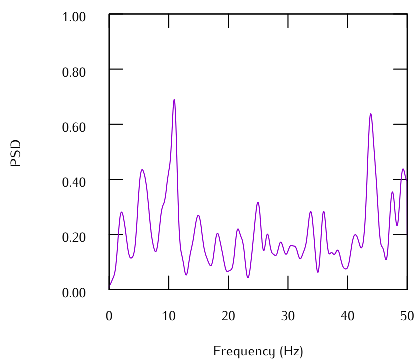


- As expected increasing `nFFT` does not help.

.33

pwelch Function - cos + WGN signal

```
Fs = 100;   t = 0:1/Fs:5;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,128,[],4096,Fs,'onesided');
```

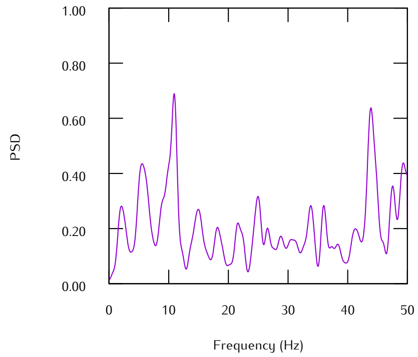


- Decreasing the window size decreases the variance.
- Still can't resolve the two frequencies.

.34

pwelch Function - cos + WGN signal

```
Fs = 100;   t = 0:1/Fs:50;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,128,[],4096,Fs,'onesided');
```

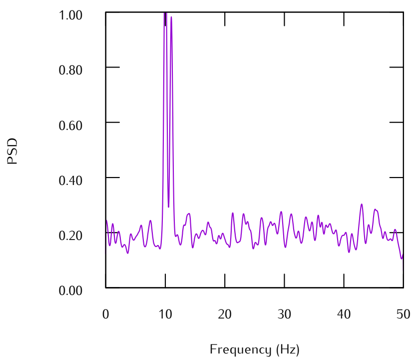


- Length of data sequence must be increased.
- Still can't resolve the two frequencies as the window size is too small.

.35

pwelch Function - cos + WGN signal

```
Fs = 100;    t = 0:1/Fs:50;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,256,[],4096,Fs,'onesided');
```



- Now we can resolve the two frequencies.

.36

Spectral Estimation - Remarks

- The length of the data sequence determines the maximum resolution that can be observed.
- Increasing the window length of each segment in the data increases the resolution.
- Decreasing the window length of each segment in the data decreases the variance of the estimate.
- `nFFT` only affects the amount of details shown and not the resolution.

.37