EE 341 – Homework Chapter 2

2.1 The electrical circuit shown in Fig. P2.1 consists of two resistors R1 and R2 and a capacitor C.
(i) Determine the differential equation relating the input voltage v(t) to the output voltage y(t).

(ii) Determine whether the system is
(a) Linear, (b) time-invariant; (c) memoryless; (d) causal, (e) invertible, and (f) stable.

(i)
Applying Kirchoff's current law to node 1

$$\frac{y(t)-v(t)}{R1}+\frac{y(t)}{R2}+C\frac{dy}{dt}=0$$

$$\frac{dy}{dt}+\frac{R1+R2}{CR1R2}y(t)=\frac{v(t)}{CR1}$$

(ii) Determine whether the system is
(b) Linear
For v1(t) applied as the input, the output is y1(t), also for v2(t) applied the output is y2(t):
$CR1\frac{dy1}{dt}+\frac{R1+R2}{R2}y1(t)=v1(t)$ and $CR1\frac{dy2}{dt}+\frac{R1+R2}{R2}y2(t)=v2(t)$

Also let,
$v3(t)=\alpha v1(t)+\beta v2(t)$ or $CR1\frac{dy3}{dt}+\frac{R1+R2}{R2}y3(t)=\alpha v1(t)+\beta v2(t)$

Substituting v1(t) and v2(t) we'll have

$$CR1\frac{dy3}{dt}+\frac{R1+R2}{R2}y3(t)=\alpha[CR1\frac{dy1}{dt}+\frac{R1+R2}{R2}y1(t)]+\beta[CR1\frac{dy2}{dt}+\frac{R1+R2}{R2}y2(t)]$$

$$CR1\frac{dy3}{dt}+\frac{R1+R2}{R2}y3(t)=CR1[\frac{d(\alpha y1+\beta y2)}{dt}+\frac{R1+R2}{R2}(\alpha y1+\beta y2)]$$

So $y3(t)=\alpha y1(t)+\beta y2(t)$

System is linear.

(c) time-invariant
For v(t-t0) applied as the input, the output y1(t) is:

$$\frac{dy1}{dt}+\frac{R1+R2}{CR1R2}y1(t)=\frac{1}{CR1}v(t-t0)$$

Substitute τ=t-t0 (so dt=dτ)

$$\frac{dy1(\tau+t0)}{d\tau}+\frac{R1+R2}{CR1R2}y1(\tau+t0)=\frac{1}{CR1}v(\tau)$$

So y1(τ)=y(τ-t0)
The system is time-invariant

(d) memoryless

$$y(t) = \frac{1}{CR1}\int_{-\infty}^{t} v(\tau)d\tau - \frac{R1+R2}{CR1R2}\int_{-\infty}^{t} y(\tau)d\tau$$

The output y(t) as t =t0 is

$$y(t) = \frac{1}{CR1}\int_{-\infty}^{t0} v(\tau)d\tau - \frac{R1+R2}{CR1R2}\int_{-\infty}^{t0} y(\tau)d\tau$$

It is clear that the output depends on previous values of the input v(t).
System is not memoryless

(e) causal
System is causal since it only depends on previous inputs.

(f) invertible

$$v(t) = CR1\frac{dy}{dt} + \frac{R1+R2}{R2}y(t)$$

The system is invertible

(g) stable.
The system is BIBO stable since a bounded input will always produce a bounded output.

2.5 Eq. (2.16) describes a linear, second-order, constant-coefficient differential equation used to model a mechanical spring damper system.
(i) By expressing Eq. (2.16) in the following form:

$$\frac{d^2y}{dt^2} + \frac{\omega n}{Q}\frac{dy}{dt} + \omega n^2 y(t) = \frac{1}{M}x(t)$$

Determine the values of ωn and Q in terms of mass M, damping factor r, and the spring constant k.

(ii) The variable ωn denotes the natural frequency of the spring damper system. Show that the natural frequency ωn can be increased by increasing the value of the spring constant k or by decreasing the mass M.

(iii) Determine whether the system is
(a) Linear, (b) time-invariant; (c) memoryless; (d) causal, (e) invertible, and (f) stable.

(i) By expressing Eq. (2.16) in the following form:

$$\frac{d^2y}{dt^2} + \frac{\omega n}{Q}\frac{dy}{dt} + \omega n^2 y(t) = \frac{1}{M}x(t)$$

Determine the values of ωn and Q in terms of mass M, damping factor r, and the spring constant k.

$$M\frac{d^2y}{dt^2} + \frac{M\omega n}{Q}\frac{dy}{dt} + M\omega n^2 y(t) = x(t)$$

Comparing to Eq. 2.16 we will have:

$\omega n = \sqrt{k/M}$ and $Q = \frac{\sqrt{kM}}{r}$

(ii) The variable ωn denotes the natural frequency of the spring damper system. Show that the natural frequency ωn can be increased by increasing the value of the spring constant k or by decreasing the mass M.

From $\omega n = \sqrt{k/M}$ we can see that increasing k increases ωn, or decreasing M.

(iii) Determine whether the system is

(a) linear

$M\frac{d^2y1}{dt^2} + \frac{M\omega n}{Q}\frac{dy1}{dt} + M\omega n^2 y1(t) = x1(t)$ and $M\frac{d^2y2}{dt^2} + \frac{M\omega n}{Q}\frac{dy2}{dt} + M\omega n^2 y2(t) = x2(t)$

$x3(t) = \alpha x1(t) + \beta x2(t)$ so $M\frac{d^2y3}{dt^2} + \frac{M\omega n}{Q}\frac{dy3}{dt} + M\omega n^2 y3(t) = \alpha x1(t) + \beta x2(t)$

$$M\frac{d^2y3}{dt^2} + \frac{M\omega n}{Q}\frac{dy3}{dt} + M\omega n^2 y3(t)$$
$$= \alpha[M\frac{d^2y1}{dt^2} + \frac{M\omega n}{Q}\frac{dy1}{dt}] + \beta[M\frac{d^2y2}{dt^2} + \frac{M\omega n}{Q}\frac{dy2}{dt} + M\omega n^2 y2(t)]$$

So y3(t)=αy1(t)+βy2(t)

The system is linear.

(b) time-invariant

For x(t-t0) applied as the input, the output y1(t) is:

$$M\frac{d^2y1}{dt^2} + \frac{M\omega n}{Q}\frac{dy1}{dt} + M\omega n^2 y1(t) = x(t)$$

Substitute τ=t-t0 so dt=dτ

$$M\frac{d^2y1(\tau+t0)}{dt^2} + \frac{M\omega n}{Q}\frac{dy1(\tau+t0)}{dt} + M\omega n^2 y1(\tau+t0) = x(t+t0)$$

So y1(τ)=y(τ-t0)

The system is time-invariant.

(c) memoryless

To get the output we need to integrate twice the second order diff. eqn.

$$y(t) = \frac{1}{M}\int_{-\infty}^{t}\int_{-\infty}^{\tau} x(\alpha)d\alpha d\tau - \omega n^2 \int_{-\infty}^{t}\int_{-\infty}^{\tau} y(\alpha)d\alpha d\tau - \frac{\omega n}{Q}\int_{-\infty}^{t} y1(\alpha)d\alpha$$

It is clear that the output depends on past values of the input.
System is not memoryless.

(d) causal
From part © we can deduce that the system is causal

(e) invertible
x(t) can be obtained from equation above so system is invertible.

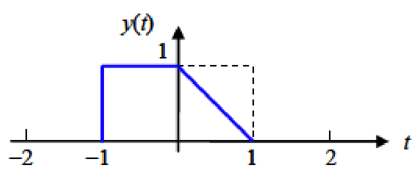
(f) stable.
The system is BIBO.

2.11 For an LTIC system, an input x(t) produces an output y(t) as shown in Fig. P2.11. Sketch the outputs for the following set of inputs:

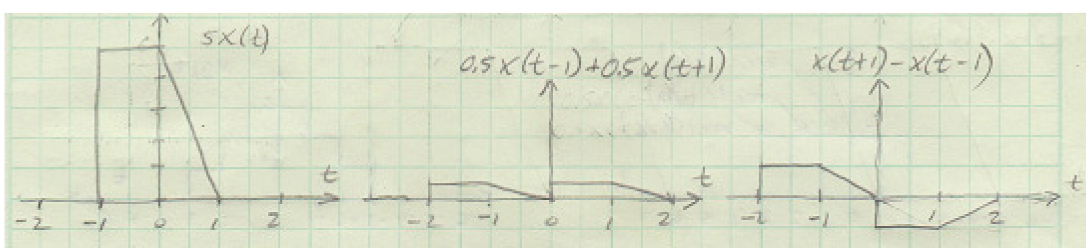
(i) 5x(t)

(ii) 0.5x(t-1)+0.5x(t+1)

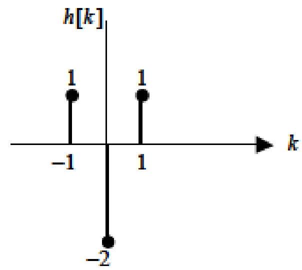
(iii) x(t+1)-x(t-1)



(i) Using linearity property 5x(t) → 5y(t)
(ii) Using linearity property 0.5x(t-1)+ 0.5x(t+1) → 0.5y(t-1)+ 0.5y(t+1)
(iii) Using linearity property x(t-1)-x(t+1) → y(t-1)-y(t+1)



2.18 The output h[k] of a DT LTI system in response to a unit impulse function δ[k] is shown in Fig. P2.18. Find the output for the following set of inputs:



(i) x[k]=δ[k+1]+δ[k]+δ[k-1]

(ii) x[k]=$\sum_{m=-\infty}^{\infty} \delta[k-4m]$

(iii) x[k]=u[k]

(i) x[k]=δ[k+1]+δ[k]+δ[k-1]

The impulse response is given by δ[k] → h[k]=δ[k+1]-2δ[k]+δ[k-1]

So the response to input x[k] is given by

δ[k+1] → h[k+1]=δ[k+2]-2δ[k+1]+δ[k]

δ[k] → h[k]=δ[k+1]-2δ[k]+δ[k-1]

δ[k-1] → h[k-1]=δ[k]-2δ[k-1]+δ[k-2]

So the total response is:

δ[k+1]-2δ[k]+δ[k-1] → δ[k+2]-δ[k+1]-δ[k-1]+]+δ[k-2]

(ii) x[k]=$\sum_{m=-\infty}^{\infty} \delta[k-4m]$

The response will be a periodic signal with period K=4. One period is given by;

$$y[k] = \begin{cases} 1 & k = -1 \\ -2 & k = 0 \\ 1 & k = 1 \\ 0 & k = 2 \end{cases}$$

(iii) x[k]=u[k]

The response will be a causal signal (due to a unit step):

y[k]=h[k]+h[k-1]+h[k-2]+….

2.21 The series and parallel configurations of systems S1 and S2 are shown in Fig. P2.21. The two systems are specified by the following input-output relationships:

S1: y[k]=x[k]-2x[k-1]+x[k-2]

S2: y[k]=x[k]+x[k-1]-2x[k-2]

(i) Show that S1 and S2 are LTI systems

(ii) Calculate the input-output relationship for the series configuration of systems S1 and S2 as shown in Fig. P2.21(a).

(iii) Calculate the input-output relationship for the parallel configuration of systems S1 and S2 as shown in Fig. P2.21(b).

(iv) Show that the series and parallel configurations of systems D1 and S2 are LTI systems.

(i) Show that S1 and S2 are LTI systems

Use the property of linear, constant coefficient finite difference equation which represents a LTI system

(ii) Calculate the input-output relationship for the series configuration of systems S1 and S2 as shown in Fig. P2.21(a).

Denote the output of S1 as:

$$w[k]=x[k]-2x[k-1]+x[k-2]$$

Denoting the output of S2 as y[k] (given that the input is w[k]):

$$y[k]=w[k]+w[k-1]-2w[k-2]$$

Substituting the first equation into the second one we'll have:

$$y[k]=(x[k]-2x[k-1]+x[k-2])+(x[k-1]-2x[k-2]+x[k-3])-2(x[k-2]-2x[k-3]+x[k-4])$$

So

$$y[k]=x[k]-x[k-1]-3x[k-2])+5x[k-3]-2x[k-4]$$

(iii) Calculate the input-output relationship for the parallel configuration of systems S1 and S2 as shown in Fig. P2.21(b).

The outputs of systems in parallel are added:

$$y[k]=2x[k]-x[k-1]-x[k-2]$$

(iv) Show that the series and parallel configurations of systems D1 and S2 are LTI systems.

Since both represent linear, constant coefficient differential equations then both systems are LTI systems.