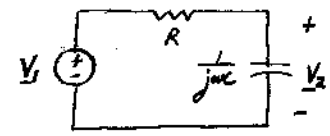
Problems 10.1, 10.4, 10.5, 10.7

PR10.1



$$\underline{V}_2 = \underline{V}_1 \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{\underline{V}_1}{j\omega CR + 1} \Rightarrow H(j\omega) = \frac{\underline{V}_2}{\underline{V}_1} = \frac{1}{1 + j\omega CR}$$

*amp. response: $|H| = \frac{1}{\sqrt{1^2 + \omega^2 C^2 R^2}}$

→ @ $\omega = 0$: $|H| = 1$ which is max

@ $\omega \to \infty$: $|H| \to 0$

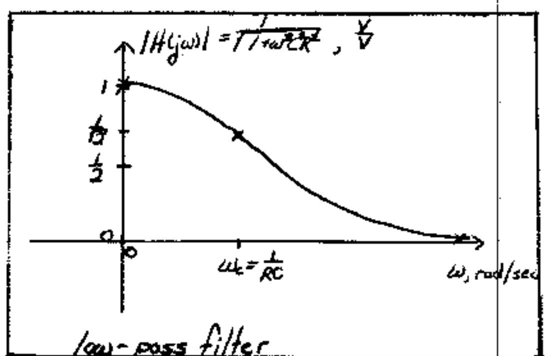
@ $\omega = \omega_c$: $|H(j\omega_c)| = \frac{|H|_{max}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1 + \omega_c^2 C^2 R^2}}$

$\Rightarrow \frac{1}{2} = \frac{1}{1 + \omega_c^2 R^2 C^2}$

$\Rightarrow 1 + \omega_c^2 R^2 C^2 = 2$

$\Rightarrow \omega_c^2 = \frac{1}{R^2 C^2}$

$\Rightarrow \omega_c = \oplus \frac{1}{RC}$



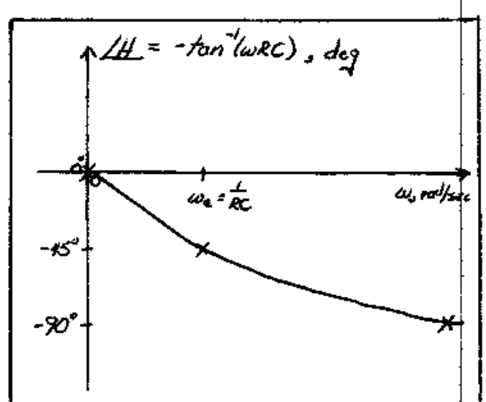
*phase response:

$$\angle H = 0 - \tan^{-1}\left(\frac{\omega CR}{1}\right) = -\tan^{-1}(\omega CR)$$

→ @ $\omega = 0$: $\angle H = 0$

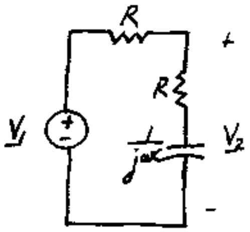
@ $\omega \to \infty$: $\angle H \to -90°$

@ $\omega = \omega_c = \frac{1}{RC}$: $\angle H = -\tan((\frac{1}{RC})CR) = -45°$



PR10.4

a)



$$\underline{V}_2 = \frac{\underline{V}_1\left(R + \frac{1}{j\omega C}\right)}{R + R + \frac{1}{j\omega C}} = \frac{\underline{V}_1 (j\omega CR + 1)}{j\omega C2R + 1} \Rightarrow H(j\omega) = \frac{1 + j\omega CR}{1 + j\omega C2R}$$

$$\Rightarrow |H| = \frac{\sqrt{1 + (\omega CR)^2}}{\sqrt{1 + (\omega C2R)^2}}$$

$\rightarrow$ find $|H|^2_{max}$: $\frac{d|H|^2}{d\omega} = \frac{d}{d\omega}\left(\frac{1 + (\omega CR)^2}{1 + (\omega C2R)^2}\right)$:

$$= \frac{2\omega C^2R^2(1 + (\omega C2R)^2) - (1 + (\omega CR)^2)(2\omega C^2 2^2 R^2)}{(1 + (\omega C2R)^2)^2}$$

$$= \frac{2\omega C^2R^2 + 2\omega C^2R^2(\omega C2R)^2 - 8\omega C^2R^2 - 8\omega C^2R^2(\omega CR)^2}{(1 + (\omega C2R)^2)^2}$$

$$= \frac{-6\omega C^2R^2}{(1 + (\omega C2R)^2)^2} = 0$$

$\Rightarrow |H|^2 =$ max when $\omega = 0$, min when $\omega \to \infty$

$\Rightarrow |H| =$ max when $\omega = 0$

$$\Rightarrow |H|_{max} = 1$$

$\rightarrow$ find ω_c : $|H(j\omega_c)|^2 = \frac{1}{2} = \frac{1 + \omega_c^2 C^2 R^2}{1 + 4\omega_c^2 C^2 R^2}$

$$\Rightarrow 1 + 4\omega_c^2 C^2 R^2 = 2 + 2\omega_c^2 C^2 R^2$$

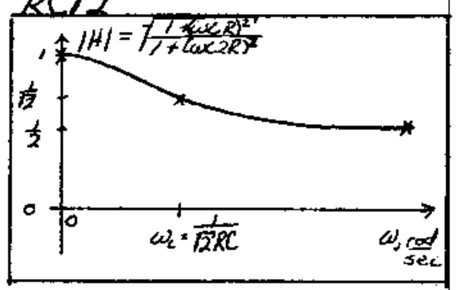
$$\Rightarrow 2\omega_c^2 C^2 R^2 = 1$$

$$\Rightarrow \omega_c = \pm\sqrt{\frac{1}{2C^2R^2}} = \oplus \frac{1}{RC\sqrt{2}}$$

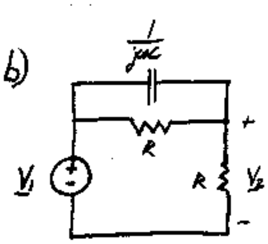
$\rightarrow$ @ $\omega = 0$: $|H| = 1$

@ $\omega = \omega_c = \frac{1}{\sqrt{2}RC}$: $|H| = \frac{1}{\sqrt{2}}$

@ $\omega \to \infty$: $|H| = \frac{1}{2}$



b)



$$\underline{V}_2 = \frac{\underline{V}_1 R}{\left(R + \frac{\frac{R}{j\omega C}}{R + \frac{1}{j\omega C}}\right)} = \frac{\underline{V}_1 R}{R + \frac{R}{j\omega CR + 1}}$$

$$= \frac{\underline{V}_1 (j\omega CR + 1)R}{(j\omega CR + 1)R + R} \Rightarrow H(j\omega) = \frac{\underline{V}_2}{\underline{V}_1} = \frac{j\omega CR^2 + R}{j\omega CR^2 + 2R}$$

$$\Rightarrow |H| = \frac{\sqrt{R^2 + \omega^2 C^2 R^4}}{\sqrt{4R^2 + \omega^2 C^2 R^4}}$$

which is min @ $\omega = 0$, max @ $\omega \to \infty$

$\rightarrow$ @ $\omega = 0$: $|H| = \frac{1}{2}$ which is min

@ $\omega \to \infty$: $|H| = 1$ which is max

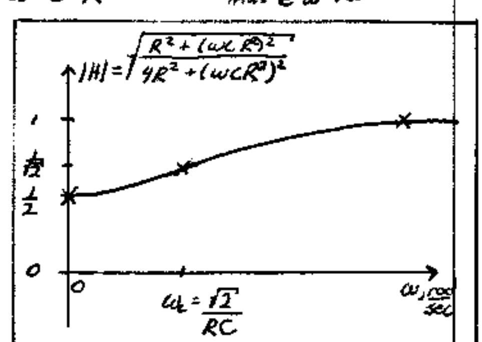
@ $\omega = \omega_c$: $|H|^2 = \frac{1}{2} = \frac{R^2 + \omega_c^2 C^2 R^4}{4R^2 + \omega_c^2 C^2 R^4}$

$$\Rightarrow 4R^2 + \omega_c^2 C^2 R^4 = 2(R^2 + \omega_c^2 C^2 R^4)$$

$$\Rightarrow 2R^2 = \omega_c^2 C^2 R^4$$

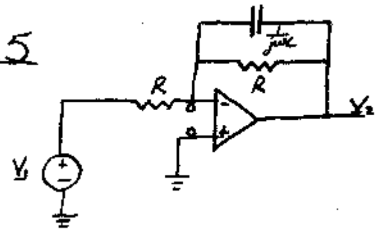
$$\Rightarrow \frac{2}{C^2R^2} = \omega_c^2$$

$$\Rightarrow \omega_c = \oplus \frac{\sqrt{2}}{RC}$$



PR 10.5

a)



$e^{1\text{-}1}$: $\frac{0-V_1}{R} + \frac{0-V_2}{\frac{1}{j\omega C}} + \frac{0-V_2}{R} = 0$

$\Rightarrow -V_1 - j\omega C R V_2 - V_2 = 0$

$\Rightarrow V_2(1 + j\omega C R) = -V_1$

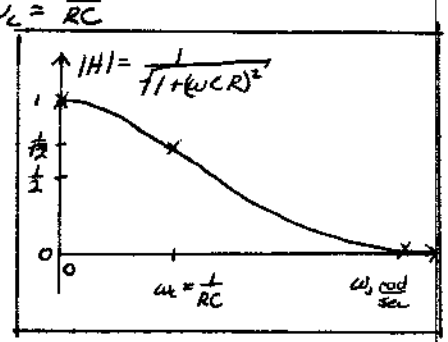
$\Rightarrow H(j\omega) = \frac{V_2}{V_1} = \frac{-1}{1 + j\omega C R}$

$\Rightarrow |H(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 C^2 R^2}}$

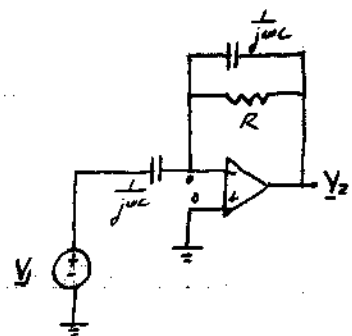
@ $\omega = 0$, $|H| = 1$

@ $\omega \to \infty$, $|H| \to 0$

@ $\omega = \omega_c$, $|H|^2 = \frac{1}{2} = \frac{1}{1 + \omega_c^2 C^2 R^2}$ $\Rightarrow 1 + \omega_c^2 C^2 R^2 = 2$ $\Rightarrow \omega_c = \frac{1}{RC}$



b)



$e^{1\text{-}1}$: $\frac{0-V_1}{\frac{1}{j\omega C}} + \frac{0-V_2}{\frac{1}{j\omega C}} + \frac{0-V_2}{R} = 0$

$\Rightarrow -j\omega C R V_1 - j\omega C R V_2 - V_2 = 0$

$\Rightarrow H(j\omega) = \frac{V_2}{V_1} = \frac{-j\omega C R}{1 + j\omega C R}$

$\Rightarrow |H| = \frac{\omega C R}{\sqrt{1 + (\omega C R)^2}}$ which is min @ $\omega = 0$, max @ $\omega \to \infty$

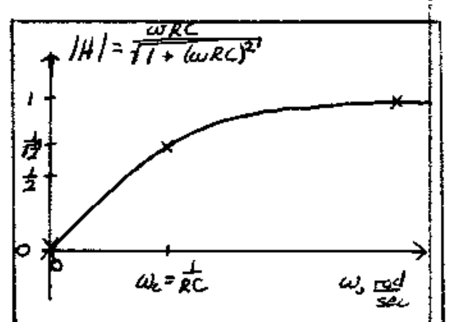
@ $\omega = 0$, $|H| = 0$

@ $\omega \to \infty$, $|H| \to 1$

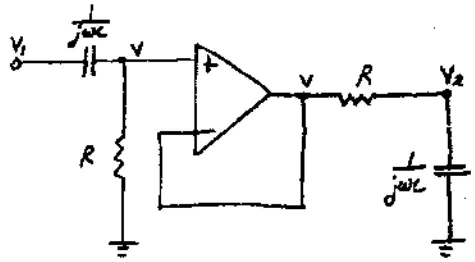
@ $\omega = \omega_c$, $|H|^2 = \frac{1}{2} = \frac{\omega_c^2 R^2 C^2}{1 + \omega_c^2 R^2 C^2}$ $\Rightarrow 1 + \omega_c^2 R^2 C^2 = 2\omega_c^2 R^2 C^2$

$\Rightarrow 1 = \omega_c^2 R^2 C^2$

$\Rightarrow \omega_c = \cancel{\pm} \frac{1}{RC}$



10.7



@ '+': $\frac{V - V_1}{\frac{1}{j\omega C}} + \frac{V - 0}{R} = 0$

$\Rightarrow j\omega C V - j\omega C V_1 + \frac{V}{R} = 0$

$\Rightarrow V(1 + j\omega C R) = j\omega C R V_1$

@ V_2: $\frac{V_2 - V}{R} + \frac{V_2 - 0}{\frac{1}{j\omega C}} = 0$

$\Rightarrow V_2 - V + j\omega C R V_2 = 0$

$\Rightarrow V_2(1 + j\omega RC) = V$

$$= \frac{j\omega RC V_1}{1 + j\omega CR}$$

$$\Rightarrow H(j\omega) = \frac{V_2}{V_1} = \frac{j\omega RC}{(1 + j\omega CR)^2} \Rightarrow \frac{\omega RC}{1 + \omega^2 C^2 R^2} = |H(j\omega)|$$

→ find max $\frac{d|H(j\omega)|}{d\omega} = \frac{RC(1 + \omega^2 C^2 R^2) - \omega RC(2\omega C^2 R^2)}{(1 + \omega^2 C^2 R^2)^2} = 0$

$\Rightarrow RC + \omega^2 C^3 R^3 - 2\omega^2 C^3 R^3 = 0$ ← max occurs here, min when $\omega \to \infty$

$\Rightarrow RC = \omega^2 C^3 R^3$

$\Rightarrow \omega^2 = \frac{1}{C^2 R^2} \Rightarrow \omega = \frac{1}{RC}$ = freq of max

$$|H(j\tfrac{1}{RC})| = \max = \frac{1}{1+1} = \frac{1}{2}$$

→ find half-power freq, ω_c

$|H(j\omega_c)| = \frac{1}{2\sqrt{2}} = \frac{\omega_c RC}{1 + \omega_c^2 R^2 C^2}$

$\Rightarrow 1 + \omega_c^2 R^2 C^2 = \omega_c RC 2\sqrt{2}$

$\Rightarrow R^2 C^2 \omega_c^2 - RC 2\sqrt{2}\,\omega_c + 1 = 0$

$\Rightarrow \omega_c = \frac{RC2\sqrt{2} \pm \sqrt{R^2C^2 8 - 4R^2C^2}}{2R^2C^2}$

$= \frac{2\sqrt{2} \pm 2}{2RC} = \frac{\sqrt{2} \pm 1}{RC}$

$= \frac{\sqrt{2}-1}{RC}, \frac{\sqrt{2}+1}{RC}$