EE 341 - Final Exam

December 14, 2004

Name: ____________________

Closed book. Show all work. Partial credit will be given. No credit will be given if an answer appears with no supporting work. You may use one page of notes and a calculator.

1. Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period. If a signal is not periodic, explain why it isn't.

(a) $x(t) = \cos\left(t + \frac{\pi}{4}\right) + \sin\left(2t + \frac{1}{4}\right)$ $\omega_1 = 1$ $T_1 = \frac{2\pi}{\omega_1} = 2\pi$

$\omega_2 = 2$ $T_2 = \frac{2\pi}{\omega_2} = \pi$

$\frac{T_1}{T_2} = \frac{2\pi}{\pi} = \frac{2}{1} = \frac{p}{q}$ $T_0 = pT_2 = qT_1 = \boxed{2\pi \text{ sec}}$

(b) $x(t) = \cos^2\left(\frac{\pi}{8}t\right)$ $= \frac{1}{2}\left(1 + \cos\frac{\pi}{4}t\right)$ $\omega = \frac{\pi}{4}$ $T = \frac{2\pi}{\omega} = \boxed{8 \text{ sec}}$

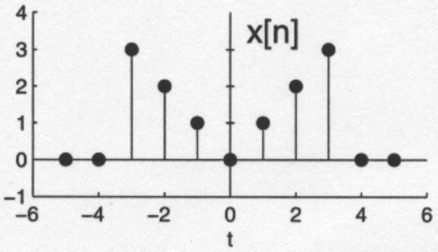
(c) $x[n] = \cos\left(\frac{\pi}{3}n\right) + \cos\left(\frac{\pi}{4}n\right)$ $\Omega_1 = \frac{\pi}{3}$ $N_1 = \frac{2\pi}{\Omega_1} = 6$

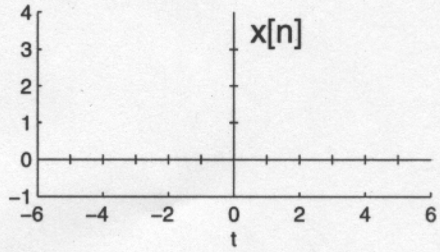
$\Omega_2 = \frac{\pi}{4}$ $N_2 = \frac{2\pi}{\omega_2} = 8$

$\frac{N_1}{N_2} = \frac{p}{q} = \frac{6}{8} = \frac{3}{4}$

$N = qN_1 = pN_2 = 3 \cdot 8 = \boxed{24}$

2. A discrete-time signal $x[n]$ is shown below. Sketch and label the signal $x[n-2]\{u(n+2)-u(n)\}$.





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3. A system may or may not be (i) Memoryless, (ii) Causal, (iii) Linear, (iv) Time Invariant. Determine which of these properties hold and which do not hold for each of the following systems. Justify your answer. In each example, $y(t)$ or $y[n]$ denotes the system output and $x(t)$ or $x[n]$ is the system input.

(i) Has memory - depends on time $t-2$

(a) $y(t) = \begin{cases} 0 & \text{if } t < 0 \\ x(t) + x(t-2) & \text{if } t \geq 0 \end{cases}$ (ii) Causal - Does not depend on future inputs

(iii) $y(t) = (x(t) + x(t-2))u(t)$

$x_1(t) \to \square \to y_1(t) = u(t)(x_1(t) + x_1(t-2))$

$x_2(t) \to \square \to y_2(t) = u(t)(x_2(t) + x_2(t-2))$

$a_1 x_1(t) + a_2 x_2(t) \to \square \to u(t)[a_1 x_1(t) + a_2 x_2(t) + a_1 x_1(t-2) + a_2 x_2(t-2)]$

$= a_1 y_1(t) + a_2 y_2(t)$

Linear

(iv) $y(t) = u(t)x(t) + u(t)x(t-2)$

Coefficient depends on time, not time invariant

(i) No (ii) Yes (iii) Yes (iv) No

(b) $y[n] - 2y[n-1] = x[n] + 2nx[n-2]$

(i) $y[n] = x[n] + 2n\,x[n-2] + 2y[n-1]$

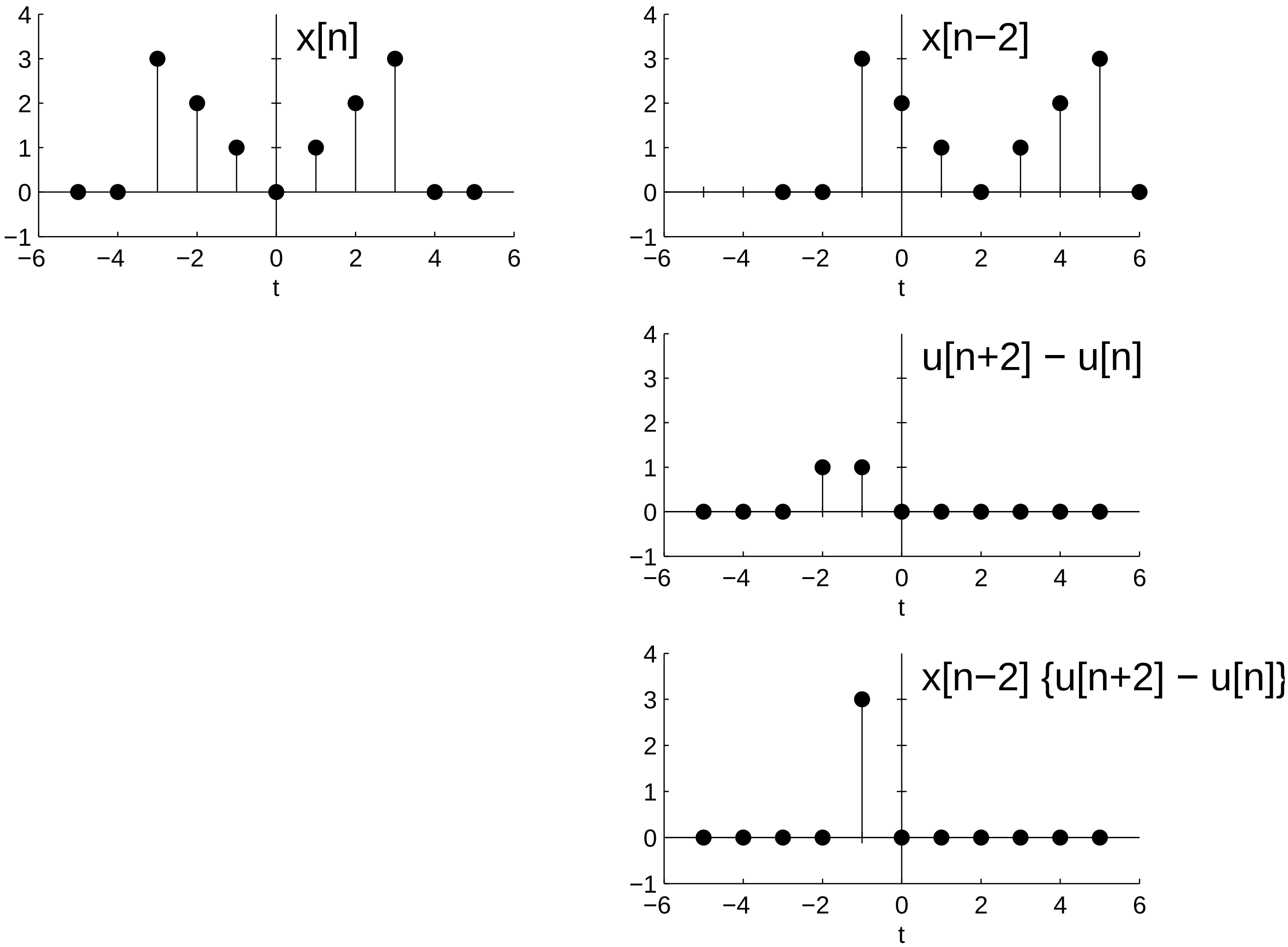
Depends on times other than n $\Rightarrow$ has memory

(ii) $y[n]$ depends only on past inputs and outputs $\Rightarrow$ causal

(iii) Is a linear difference eqn, of form of eqn 1.59 of text $\Rightarrow$ Linear

(iv) Has time varying coefficient n $\Rightarrow$ not time invariant

(i) No (ii) Yes (iii) Yes (iv) No


x[n]
t
x[n−2]
t
u[n+2] − u[n]
t
x[n−2] {u[n+2] − u[n]}
t

4. Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n],$$

with the input

$$x[n] = \left(\frac{1}{3}\right)^n u[n].$$

The initial condition is $y[-1] = -2$.

Find a closed form expression for the output $y[n]$, solving the system in the time domain.

Characteristic eqn: $\lambda - \frac{1}{2} = 0$ $\quad \lambda = \frac{1}{2}$ $\quad y_h[n] = A\lambda^n = A\left(\frac{1}{2}\right)^n$

$x[n] = \left(\frac{1}{3}\right)^n u[n] \Rightarrow y_p[n] = K\left(\frac{1}{3}\right)^n$

$y[n] = y_h[n] + y_p[n] = A\left(\frac{1}{2}\right)^n + K\left(\frac{1}{3}\right)^n$

2 unknowns A, K need 2 eqns

$y[0] = A\left(\frac{1}{2}\right)^0 + K\left(\frac{1}{3}\right)^0 = A + K$

$y[1] = A\left(\frac{1}{2}\right)^1 + K\left(\frac{1}{3}\right)^1 = \frac{1}{2}A + \frac{1}{3}K$

$y[0] = x[0] + \frac{1}{2}y[-1] = \left(\frac{1}{3}\right)^0 u[0] + \frac{1}{2}(-2) = 1 - 1 = 0$

$y[1] = x[1] + \frac{1}{2}y[0] = \left(\frac{1}{3}\right)^1 u[1] + \frac{1}{2}(0) = \frac{1}{3}$

$A + K = 0 \Rightarrow K = -A$

$\frac{1}{2}A + \frac{1}{3}K = \frac{1}{3} \Rightarrow \frac{1}{2}A + \frac{1}{3}(-A) = \frac{1}{3} \Rightarrow A\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3} \Rightarrow \frac{1}{6}A = \frac{1}{3} \Rightarrow A = 2$

$K = -A = -2$

$y[n] = A\left(\frac{1}{2}\right)^n + K\left(\frac{1}{3}\right)^n = \boxed{2\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n \quad n \geq 0}$

5. Determine the following convolutions:

(a) $x[n] = \delta(n) + 2\delta(n-1) + 3\delta(n-2)$, and $v[n] = \delta(n) - 2\delta(n-2)$. Find the linear convolution of $x[n]$ and $v[n]$. $x(n) = \{1, 2, 3\}$ $v(n) = \{1, 0, -2\}$

Flip $v(n)$,

		1	2	3	
-2	0	1			
0	0	1	0	0	

add $\underbrace{0 \;\; 0 \;\; 1 \;\; 0 \;\; 0}_{1}$ $y(0)$

$\begin{array}{ccc} 1 & 2 & 3 \\ -2\;\;0 & 1 & \\ \hline 0 & 2 & \end{array}$ $\underbrace{0 \;\; 2}_{2}$ $y(1)$

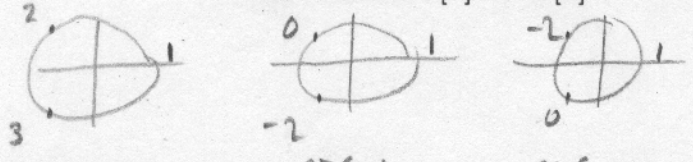
$\begin{array}{ccc} 1 & 2 & 3 \\ -2 & 0 & 1 \\ \hline -2 & 0 & 3 \end{array}$ $\underbrace{-2 \;\; 0 \;\; 3}_{1}$ $y(2)$

$\begin{array}{cccc} 1 & 2 & 3 & \\ & -2 & 0 & 1 \\ \hline & -4 & 0 & \end{array}$ -4 $y(3)$

$\begin{array}{cccc} 1 & 2 & 3 & \\ & & -2\;\;0 & 1 \\ \hline & & -6 & \end{array}$ $y(4)$

$y(n) = \{1, 2, 1, -4, -6\}$

(b) $x[n] = \delta(n)+2\delta(n-1)+3\delta(n-2)$, and $v[n] = \delta(n)-2\delta(n-2)$. Find the 3-point circular convolution of $x[n]$ and $v[n]$.



$x(n)$ $v(n)$ $v(-n)$

$v(-(n+1))$

$$\boxed{y = [-3, -4, 1]}$$

To get $y(0)$, put $v(-n)$ over $x(n)$, multiply and add

$y(0) = (1)(1) + (2)(-2) + (3)(0) = -3$

To get $y(1)$ put $v[-(n+1)]$ over $x(n)$

$y(1) = (1)(0) + (2)(1) + (3)(-2) = -4$

To get $y(2)$ put $v[-(n+2)]$ over $x(n)$

$y(2) = (1)(-2) + (2)(0) + (3)(1) = 1$

(c) $x[n] = (\frac{1}{2})^n u[n]$ and $h[n] = u[n+2]$. Find the linear convolution of $x[n]$ and $h[n]$.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) = \sum_{k=-\infty}^{\infty} u[k+2] \left(\tfrac{1}{2}\right)^{n-k} u(n-k)$$

$u[k+2] = 0$ for $k < -2$, so $y[n] = \sum_{k=-2}^{\infty} \left(\tfrac{1}{2}\right)^{n-k} u(n-k)$

$u[n-k] = 0$ for $n < k$, so $y[n] = \sum_{k=-2}^{n} \left(\tfrac{1}{2}\right)^{n-k}$

If $n < -2$, no terms in sum, and $y[n] = 0$

For $n \geq -2$, $y(n) = \left(\tfrac{1}{2}\right)^n \sum_{k=-2}^{n} \left(\tfrac{1}{2}\right)^{-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=-2}^{n} 2^k = \left(\tfrac{1}{2}\right)^n \left(\dfrac{2^{-2} - 2^{n+1}}{1-2}\right)$

$y(n) = \dfrac{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^n - 2^{n+1}\left(\frac{1}{2}\right)^n}{-1} = \left(2 - \left(\tfrac{1}{2}\right)^{n+2}\right)$ $n \geq -2$

$$\boxed{y(n) = \left[2 - \left(\tfrac{1}{2}\right)^{n+2}\right] u[n+2]}$$

6. Find the Fourier series representation of the continuous-time periodic signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

$\omega_1 = \frac{2\pi}{3}$ $\quad T_1 = \frac{2\pi}{\omega_1} = 3$ $\qquad \omega_2 = \frac{5\pi}{3}$ $\quad T_2 = \frac{2\pi}{\omega_2} = \frac{6}{5}$

$\frac{T_1}{T_2} = \frac{3}{6/5} = \frac{5}{2} = \frac{p}{q}$ $\qquad T_0 = pT_2 = qT_1 = 5\left(\frac{6}{5}\right) = 6$

$\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{3}$

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

$$= 2 + \frac{e^{\frac{2\pi}{3}t} + e^{-\frac{2\pi}{3}t}}{2} + 4\,\frac{e^{\frac{5\pi}{3}t} - e^{-\frac{5\pi}{3}t}}{2j}$$

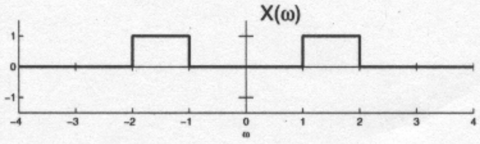
$$= 2 + \frac{1}{2}e^{\frac{2\pi}{3}t} + \frac{1}{2}e^{-\frac{2\pi}{3}t} - 2je^{\frac{5\pi}{3}t} + 2je^{-\frac{5\pi}{3}t}$$

$$\boxed{x(t) = 2 + \frac{1}{2}e^{2\omega_0 t} + \frac{1}{2}e^{-2\omega_0 t} - 2je^{5\omega_0 t} + 2je^{-5\omega_0 t}}$$

$$\boxed{c_0 = 2 \quad c_2 = \frac{1}{2} \quad c_{-2} = \frac{1}{2} \quad c_5 = -2j \quad c_{-5} = 2j}$$

7. Below are plots of the Fourier transforms $X(\omega)$ of some continuous-time signals $x(t)$. Answer the questions about the signals $x(t)$. Be sure to explain your answers.

(a)



i. Is $x(t)$ real, imaginary, or complex?

$X(\omega)$ even $\Rightarrow$ $x(t)$ real

ii. Is $x(t)$ even or odd?

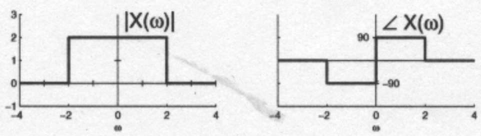
$X(\omega)$ real $\Rightarrow$ $x(t)$ even

iii. Find $x(0)$.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega = \frac{1}{2\pi}(\text{Area under curve}) = \frac{1}{2\pi}(2) = \frac{1}{\pi}$$

(b)



i. Is $x(t)$ real, imaginary, or complex?

$|X(\omega)|$ even
$\measuredangle X(\omega)$ odd
$\Big\} \Rightarrow$ $x(t)$ real

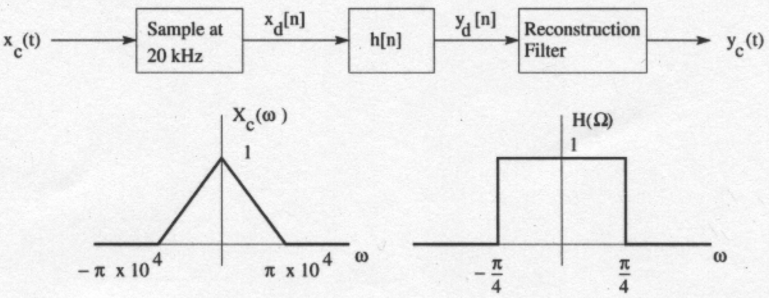
ii. Is $x(t)$ even or odd?

$X(\omega)$ purely imaginary $\Rightarrow$ $x(t)$ odd

iii. Find $x(0)$.

$x(0) = 0$ for all odd functions

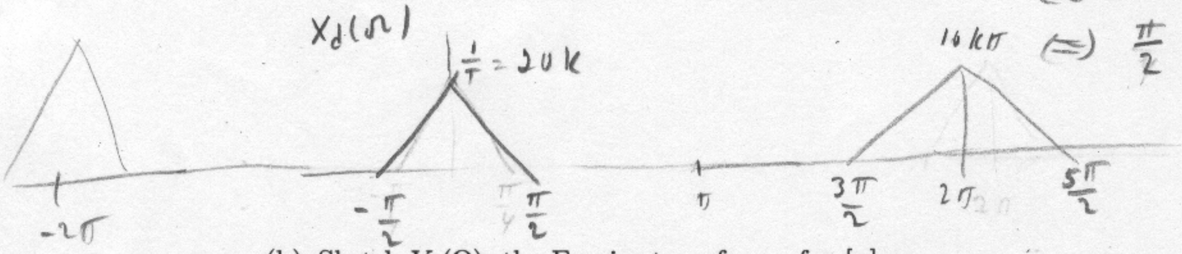
8. The figure below shows a system for filtering a continuous-time signal using a discrete-time filter. Also shown are the Fourier transform $X_c(\omega)$ of the input signal $x_c(t)$ and the frequency response $H(\Omega)$ of the discrete-time filter $h[n]$.



(a) Sketch $X_d(\Omega)$, the Fourier transform of $x_d[n]$.

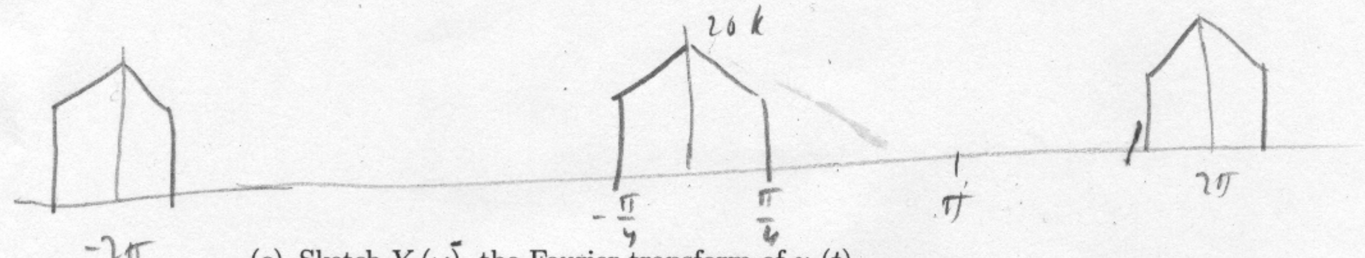
$f_s = 20\ kHz \qquad \omega_s = 2\pi f_s = 40k\pi$

CT		DT
ω_s	$\Leftrightarrow$	2π
$40k\pi$	$\Leftrightarrow$	2π
$10k\pi$	$\Leftrightarrow$	$\frac{\pi}{2}$



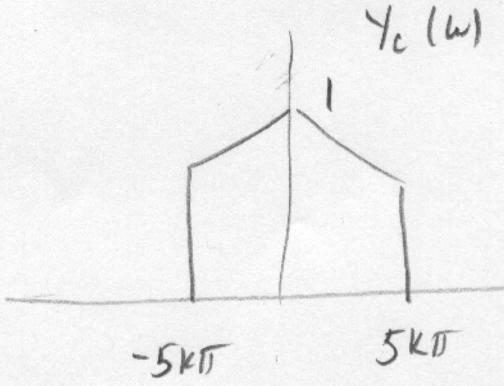
(b) Sketch $Y_d(\Omega)$, the Fourier transform of $y_d[n]$.

$H(\Omega)$ will keep from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$, stop $\frac{\pi}{4}$ to π and $-\frac{\pi}{4}$ to $-\pi$



(c) Sketch $Y_c(\omega)$, the Fourier transform of $y_c(t)$.

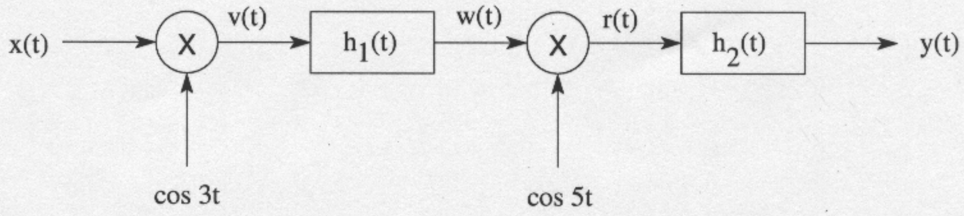
CT		DT
$5k\pi$	$\Leftrightarrow$	$\frac{\pi}{4}$

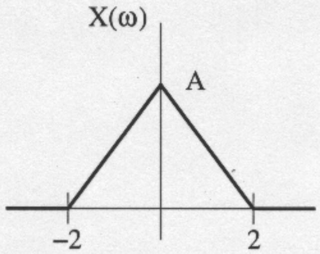


9. Consider the system below. The frequency responses of the filters are given by

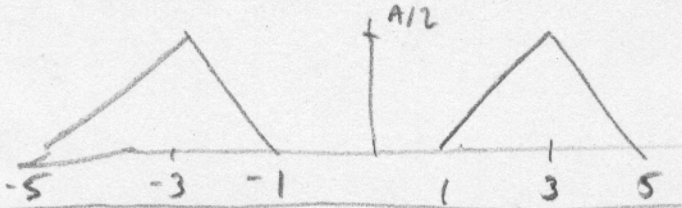
$$H_1(\omega) = \begin{cases} 0, & |\omega| < 3 \\ 1, & |\omega| \geq 3 \end{cases} \qquad H_2(\omega) = \begin{cases} 1, & |\omega| < 3 \\ 0, & |\omega| \geq 3 \end{cases}$$

The input $x(t)$ has the spectrum $X(\omega)$.

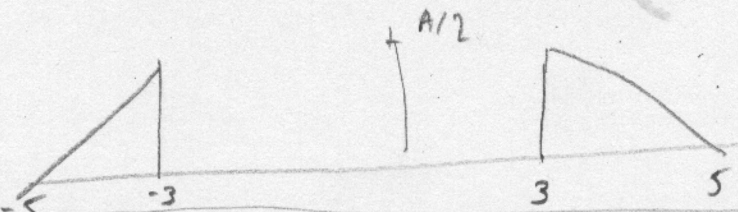




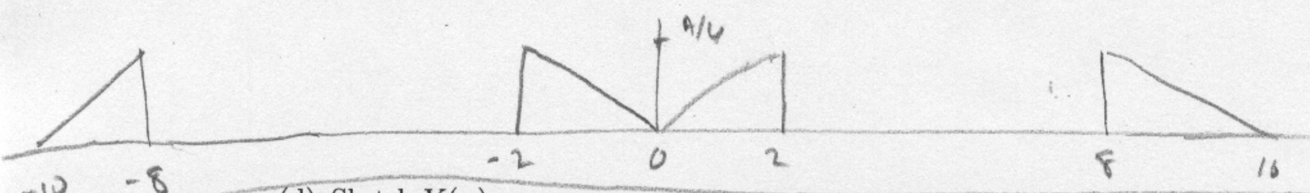
(a) Sketch $V(\omega)$ $= \frac{1}{2}\left(X(\omega+3) + X(\omega-3)\right)$



(b) Sketch $W(\omega)$ Keep $\omega > 3$, $\omega < -3$



(c) Sketch $R(\omega)$ $= \frac{1}{2}\left(W(\omega+5) + W(\omega-5)\right)$



(d) Sketch $Y(\omega)$ Keep $-3 \leq \omega \leq 3$

10. Find the following Fourier transforms:

(a) $x[n] = n\left(\frac{1}{4}\right)^n u[n]$

$$\left(\tfrac{1}{4}\right)^n u(n) \Leftrightarrow \frac{1}{1-\frac{1}{4}e^{-j\Omega}}$$

$$n\,v(n) \Leftrightarrow j\frac{d}{d\Omega}V(\Omega) = j\frac{d}{d\Omega}\left(\frac{1}{1-\frac{1}{4}e^{-j\Omega}}\right) = j(-1)\left(1-\tfrac{1}{4}e^{-j\Omega}\right)^2\left(j\tfrac{1}{4}e^{-j\Omega}\right)$$

$$= \boxed{\frac{\frac{1}{4}e^{-j\Omega}}{\left(1-\frac{1}{4}e^{-j\Omega}\right)^2}}$$

(b) $x[n] = \left(-\frac{1}{2}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n]$

$$\left(-\tfrac{1}{2}\right)^n u[n] \Leftrightarrow \frac{1}{1+\frac{1}{2}e^{-j\Omega}}$$

$$\sin\left(\tfrac{\pi}{4}n\right) v(n) \Leftrightarrow \frac{j}{2}\left[V\left(\Omega+\tfrac{\pi}{4}\right) - V\left(\Omega-\tfrac{\pi}{4}\right)\right]$$

$$= \boxed{\frac{j}{2}\left[\frac{1}{\left(1+\frac{1}{2}e^{-j\left(\Omega+\frac{\pi}{4}\right)}\right)} - \frac{1}{\left(1+\frac{1}{2}e^{-j\left(\Omega-\frac{\pi}{4}\right)}\right)}\right]}$$