EE 341

Solution of Linear Constant Coefficient Differential Equations in the Time Domain

Consider the differential equation

$$y^{(N)}(t) + \sum_{i=0}^{N-1} a_i y^{(i)}(t) = \sum_{i=0}^{M} b_i x^{(i)}(t) \tag{1}$$

where $M \leq N$.

This equation has the solution:

$$y(t) = \underbrace{\left\{ A_1 e^{s_1 t} + \cdots + A_N e^{s_N t} \right\}}_{y_{tr}(t)} + \underbrace{y_p(t)}_{y_{ss}(t)} \tag{2}$$

where the s_i's are the roots of the characteristic polynomial of the system:

$$s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0 = 0.$$

(For repeated roots, use $e^{s_i t}$, $te^{s_i t}$, $t^2 e^{s_i t}$, etc. For example, if $s_1 = s_2$, you would use $A_1 e^{s_1 t} + A_2 t e^{s_1 t}$ instead of $A_1 e^{s_1 t} + A_2 e^{s_1 t}$.)

$y_p(t)$ is the particular (steady-state) solution which depends on the input:

$x(t)$	$y_p(t)$
$Cu(t)$	$Ku(t)$
$Ce^{at}u(t)$	$Ke^{at}u(t)$
$C\cos(\omega_o t) + D\sin(\omega_o t)$	$K_1\cos(\omega_o t) + K_2\sin(\omega_o t)$

If the characteristic polynomial has a root at the value of an input exponential (e.g., $s_1 = -2$ and $x(t) = e^{-2t}$) you would use $Cte^{s_k t}$ for $y_p(t)$ (e.g., $y_p(t) = Cte^{-2t}$).

Solve for the unknowns by using the $N-1$ initial conditions and Equation (1).

The *transient response* of the system is $y_{tr}(t)$ – it is the part of the response which dies out in time. The *steady-state response* of the system is $y_{ss}(t) = y_p(t)$ – it stays around as long as the input drives it. The total response is the sum of the two: $y(t) = y_{tr}(t) + y_{ss}(t)$.

You can solve the differential equation for the case $x(t) = 0$, subject to the initial conditions of the system. This is the *natural response* or *zero-input response*, $y_{zi}(t)$, of the system. You can further solve for the case $y(0) = 0$, $y^{(1)}(0) = 0$, etc., for the actual input $x(t)$. This is the *forced response* or *zero-state response*, $y_{zs}(t)$, of the system. The total response is the sum of the two: $y(t) = y_{zi}(t) + y_{zs}(t)$.