EE 341

Solution of Difference Equations in the Time Domain

The difference equation

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

has the solution (if $M \le N$):

$$y[n] = \underbrace{\{A_1\lambda_1^n + \cdots + A_N\lambda_N^n\}}_{y_{tr}[n]} + \underbrace{y_p[n]}_{y_{ss}[n]} \tag{1}$$

where the λ_k's are the roots of the characteristic polynomial of the system:

$$\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N = 0.$$

(For repeated roots, use λ_k^n, $n\lambda_k^n$, $n^2\lambda_k^n$, etc. For example, if $\lambda_1 = \lambda_2$, you would use $A_1\lambda_1^n + A_2 n\lambda_1^n$ instead of $A_1\lambda_1^n + A_2\lambda_1^n$.)

$y_p[n]$ is the particular (steady-state) solution which depends on the input:

$x[n]$	$y_p[n]$
$C\delta[n]$	$K\delta[n]$
$Cu[n]$	K
$Cu[n]$	K
$C\alpha^n u[n]$	$K\alpha^n$
$C\cos[\omega_o n] + D\sin[\omega_o n]$	$K_1\cos[\Omega_o n] + K_2\sin[\Omega_o n]$

If the characteristic polynomial has a root at the value of an input exponential (e.g., $\lambda_1 = \frac{1}{2}$ and $x[n] = \left(\frac{1}{2}\right)^n$) you would use $Kn\lambda_k^n$ for $y_p[n]$ (e.g., $y_p[n] = Kn\left(\frac{1}{2}\right)^n$).

Solve for the unknowns by finding $y[0]$, $y[1]$, $y[2]$, $\cdots$ from the initial conditions until you get as many equations as you have unknowns. Solve these equations for the unknowns.

The *transient response* of the system is $y_{tr}[n] = A_1\lambda_1^n + \cdots + A_N\lambda_N^n + \gamma_1\delta[n] + \cdots + \gamma_{M-N}\delta[n-(M-N)]$ – it is the part of the response which dies out in time. The *steady-state response* of the system is $y_{ss}[n] = y_p[n]$ – it stays around as long as the input drives it. The total response is the sum of the two: $y[n] = y_{tr}[n] + y_{ss}[n]$.

You can solve the difference equation for the case $x[n] = 0$, subject to the initial conditions of the system. This is the *natural response* or *zero-input response*, $y_{zi}[n]$, of the system. You can further solve for the case $y[n] = 0$ for $n < 0$ for the actual input $x[n]$. This is the *forced response* or *zero-state response*, $y_{zs}[n]$, of the system. The total response is the sum of the two: $y[n] = y_{zi}[n] + y_{zs}[n]$.