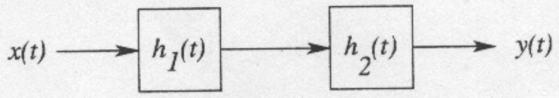
EE 341 - Final Exam

December 14, 2005

Name: Solutions

Closed book. Show all work. Partial credit will be given. No credit will be given if an answer appears with no supporting work. You may use two pages of notes and a calculator.

1. Consider the cascade system shown below:



The two systems have the impulse responses $h_1(t) = e^{-t}u(t)$ and $h_2(t) = e^{-t}u(t)$.
Find the impulse response of the total system.

$$H_1(s) = \frac{1}{s+1} \qquad H_2(s) = \frac{1}{s+1}$$

$$H(s) = H_1(s)\,H_2(s) = \frac{1}{(s+1)^2}$$

$$h(t) = te^{-t}u(t)$$

or

$$h(t) = h_1(t) * h_2(t) = \int_{-\infty}^{\infty} h_1(\lambda)\, h_2(t-\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} e^{-\lambda} u(\lambda)\, e^{-(t-\lambda)} u(t-\lambda)$$

$$= \int_0^t e^{-\lambda} e^{-(t-\lambda)}\, d\lambda = e^{-t}\int_0^t d\lambda$$

$$= te^{-t}u(t)$$

2. Find the Fourier series representation of the following signal:

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) - 4\sin\left(\frac{5\pi}{3}t\right)$$

$\omega_1 = \frac{2\pi}{3}$ $T_1 = \frac{2\pi}{\omega_1} = 3$ $\omega_2 = \frac{5\pi}{3}$ $T_2 = \frac{2\pi}{\omega_2} = \frac{6}{5}$

$\frac{T_1}{T_2} = \frac{r}{q} = \frac{3}{6/5} = \frac{15}{6} = \frac{5}{2}$ $T = q T_1 = 2 \cdot 3 = 6$ $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{3}$

$x(t) = 2 + \cos(2\omega_0 t) - 4\sin(5\omega_0 t) = 2 + \frac{1}{2}e^{2\omega_0 t} + \frac{1}{2}e^{-2\omega_0 t} - \frac{4}{2j}e^{5\omega_0 t} + \frac{4}{2j}e^{-5\omega_0 t}$

$C_0 = 2$, $C_2 = \frac{1}{2}$, $C_{-2} = \frac{1}{2}$, $C_5 = 2j$, $C_{-5} = -2j$

$C_k = 0$ for all other k

3. Let

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3] \text{ and } h[n] = 2\delta[n+1] + 2\delta[n-1]$$

Find $y[n] = x[n] * h[n]$.

$x(n) = \{\underset{\uparrow}{1}\ 2\ 0\ -1\}$ $h(n) = \{2\ \underset{\uparrow}{0}\ 2\}$

$y(n) = \sum_{-\infty}^{\infty} x(k)\,h(n-k)$

$x(k)$ $\{\underset{\uparrow}{1}\ 2\ 0\ -1\}$

$h(-k)$ $\{2\ 0\ 2\}$

$\boxed{\begin{array}{l} \text{OR} \quad H(z) = 2z + 2z^{-1}, \quad X(z) = 1 + 2z^{-1} - 3z^{-3} \\ H(z)X(z) = 2z + 4 + 2z^{-1} + 2z^{-2} - 2z^{-4} \\ y(n) = 2\delta(n+1) + 4\delta(n) + 2\delta(n-1) + 2\delta(n-2) \\ \quad - 2\delta(n-4) \end{array}}$

$y(-1)$: 1 2 0 −1 / 2 0 2 → 2

$y(0)$: 1 2 0 −1 / 2 0 2 → 4

$y(1)$: 1 2 0 −1 / 2 0 2 → 2

$y(2)$: 1 2 0 −1 / 2 0 2 → 4 −2 = 2

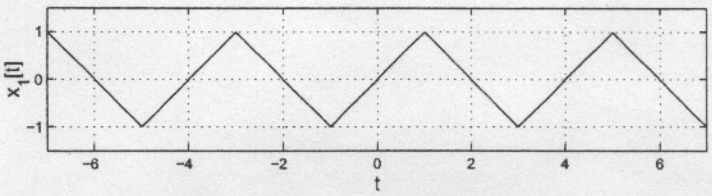
$y(3)$: 1 2 0 −1 / 2 0 2 → 0

$y(4)$: 1 2 0 −1 / 2 0 2 → −2

$y(n) = \{2\ \underset{\uparrow}{4}\ 2\ 2\ 0\ -2\}$

$y(n) = 2\delta(n+1) + 4\delta(n) + 2\delta(n-1) + 2\delta(n-2) - 2\delta(n-4)$

4. Consider the following signals:



(a) Is $X_1(\omega)$ real? Explain. No. $x_1(t)$ real, odd $\Rightarrow X_1(\omega)$ imag, odd

(b) Is $X_1(\omega)$ imaginary? Explain. Yes. see (a)

(c) Is there a real α such that $e^{j\alpha\omega}X_1(\omega)$ is real? Explain. Yes. To be real, $x(t)$ must be even. Shifting $x_1(t)$ by ± 1 sec, ± 3, ± 5, ... makes signal even. $e^{j\alpha w}X_1(w) \Leftrightarrow x_1(t+\alpha)$
$\alpha = \pm 1, \pm 3, \pm 5, \ldots$ makes $x_1(t)$ even, $X_1(w)$ real.

(d) Is $\int_{-\infty}^{\infty} X_1(\omega)d\omega = 0$? Explain. Yes

$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(w)\,dw \Rightarrow \int_{-\infty}^{\infty} X_1(w)\,dw = 2\pi x(0) = 0$

(e) Is $X_1(\omega)$ periodic? Explain.

No - $X(w)$ periodic $\Rightarrow x_1(t)$ has a δ fnc - since $x_1(t)$ does not have a δ, $X_1(w)$ is not periodic

$x_2(t) = 2\delta(t-1)$

(a) Is $X_2(\omega)$ real? Explain. No - $X_2(w) = 2e^{-jw} = 2\cos w - 2j\sin w$ is complex

(b) Is $X_2(\omega)$ imaginary? Explain. No. see (a)

(c) Is there a real α such that $e^{j\alpha\omega}X_2(\omega)$ is real? Explain. Yes:

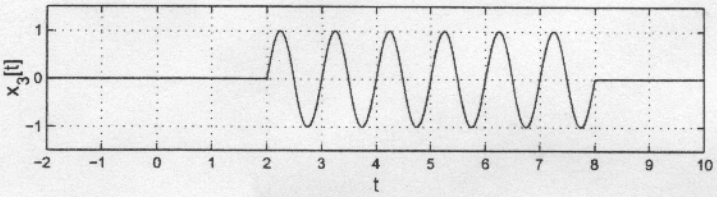
$\alpha = 1 \Rightarrow e^{j\alpha w}X_2(w) = e^{jw}\,2e^{-jw} = 2$

(d) Is $\int_{-\infty}^{\infty} X_2(\omega)d\omega = 0$? Explain. Yes

$\int_{-\infty}^{\infty} X_2(w) = 2\pi x(0) = 4\pi\,\delta(-1) = 0$

(e) Is $X_2(\omega)$ periodic? Explain.

Yes! see (a)



(a) Is $X_3(\omega)$ real? Explain. No: $X_3(t)$ neither even nor odd, so $X_3(\omega)$ is complex

(b) Is $X_3(\omega)$ imaginary? Explain. No. See (a)

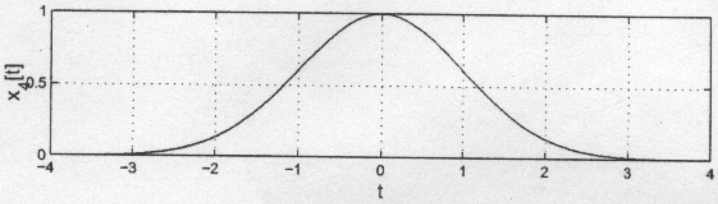
(c) Is there a real α such that $e^{j\alpha\omega}X_3(\omega)$ is real? Explain. No. No time shift makes $x_3(t)$ even

(d) Is $\int_{-\infty}^{\infty} X_3(\omega)d\omega = 0$? Explain. Yes.

$$\int_{-\infty}^{\infty} X_3(\omega) = 2\pi x_3(0) = 0$$

(e) Is $X_3(\omega)$ periodic? Explain. No

$x_3(t)$ does not have a δ



(a) Is $X_4(\omega)$ real? Explain. Yes. $x_4(t)$ real & even $\Rightarrow$ $X_4(\omega)$ real & even

(b) Is $X_4(\omega)$ imaginary? Explain. No. See (a)

(c) Is there a real α such that $e^{j\alpha\omega}X_4(\omega)$ is real? Explain. Yes. $\alpha = 0$ $- X_4(\omega)$ already real

(d) Is $\int_{-\infty}^{\infty} X_4(\omega)d\omega = 0$? Explain. No

$\int_{-\infty}^{\infty} X(\omega)\, d\omega = 2\pi\, x(0) = 2\pi$

(e) Is $X_4(\omega)$ periodic? Explain. No

$x_4(t)$ does not have δ fnc

5. Consider a system with the impulse response

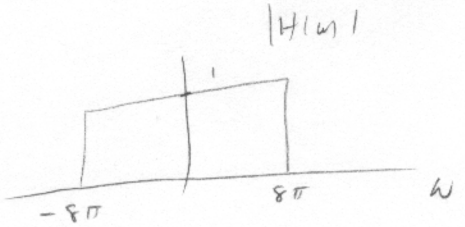
$$h(t) = 8\,\text{sinc}[8(t-1)]$$

(a) Find the frequency response $H(\omega)$ of the system.

Let $h_2(t) = 8\,\text{sinc}(8t) = \frac{1}{2\pi}\left(16\pi\,\text{sinc}\left(\frac{16\pi t}{2\pi}\right)\right) \Leftrightarrow P_{16\pi}(\omega)$

$h(t) = h_2(t-1) \Leftrightarrow e^{-j\omega} H_2(\omega)$

$H(\omega) = e^{-j\omega} P_{16\pi}(\omega)$



(b) Find the output $y(t)$ of the system when the input is $x(t) = \cos(\pi t)$.

$x(t) = \cos(\pi t)$

$\omega = \pi$

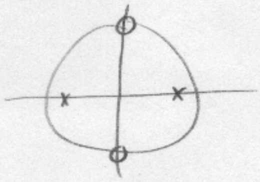
$H(\pi) = e^{-j\pi} P_{16\pi}(\pi) = e^{-j\pi}$ -1

$y(t) = |H(\pi)| \cos(\pi t + \measuredangle H(\pi))$

$= \cos(\pi t - \pi) = -\cos(\pi t)$

6. The transfer function of a causal discrete-time system has poles at $z = +0.8$ and $z = -0.8$. It has zeros at $z = j$ and $z = -j$. It is known that the gain of the system at $\Omega = 0$ is 2 — i.e., $|H(\Omega)|_{\Omega=0} = 2$.

(a) Sketch the pole-zero diagram of the system.



(b) Find the transfer function $H(z)$ for the system.

$$H(z) = G\,\frac{(z-z_1)(z-z_2)}{(z-P_1)(z-P_2)} = G\,\frac{(z-j)(z+j)}{(z-0.8)(z+0.8)} = G\,\frac{z^2+1}{z^2-0.64}$$

$$\Omega = 0 \Rightarrow z = e^{j\Omega} = e^{0} = 1 \qquad H(0) = G\,\frac{1^2+1}{1^2-0.64} = G\,\frac{2}{0.36} = 2 \qquad G = \frac{2}{2/0.36} = 0.36$$

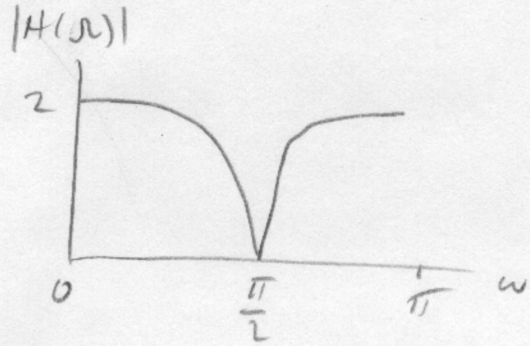
$$H(z) = 0.36\,\frac{z^2+1}{z^2-0.64}$$

(c) Write the difference equation which describes the system.

$$H(z) = 0.36\,\frac{1+z^{-2}}{1-0.64z^{-2}} = \frac{0.36+0.36z^{-2}}{1-0.64z^{-2}}$$

$$y[n] - 0.64\,y[n-2] = 0.36x[n] + 0.36x[n-2]$$

(d) Roughly sketch the gain $|H(\Omega)|$ of the system for $0 \le \Omega \le \pi$.



(e) The input to this system is $x[n] = 2\cos(\frac{\pi}{2}n) - 3\sin(\pi n)$. What is the steady-state output $y[n]$? (Note: This is easily done without taking z-transforms or inverse z-transforms.)

$\Omega = \frac{\pi}{2} \Rightarrow z = e^{j\pi/2} = j \qquad H(\frac{\pi}{2}) = 0.36 \dfrac{j^2+1}{j^2 - 0.64} = 0$

$\Omega = \pi \Rightarrow z = e^{j\pi} = -1 \qquad H(\pi) = 0.36 \dfrac{(-1)^2+1}{(-1)^2 - 0.64} = 2$

$y_{ss}(n) = H(\frac{\pi}{2})\, 2\cos(\frac{\pi}{2}n + \measuredangle \frac{\pi}{2}) - H(\pi)\, 3\sin(\pi n + \measuredangle H(\pi))$

$= 0 - 2 \cdot 3 \sin(\pi n + 0)$

$= -6 \sin(\pi n)$

(f) Is the system stable? Explain.

Yes - all poles inside unit circle

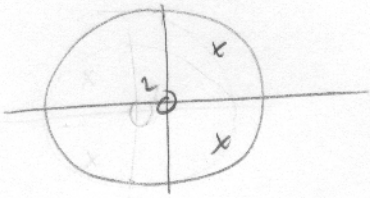
7. A discrete time system is described by the difference equation

$$y[n] - y[n-1] + 0.5y[n-2] = x[n]$$

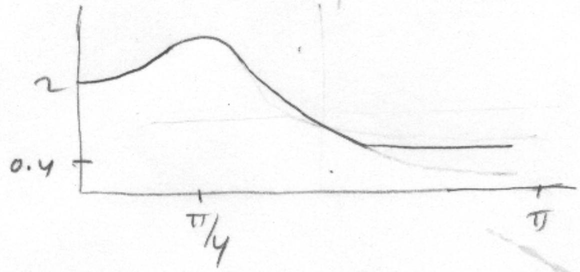
(a) Find the transfer function $H(z)$ of the system.

$$H(z) = \frac{1}{1 - z^{-1} + 0.5z^{-2}} = \frac{z^2}{z^2 - z + 0.5} = \frac{z^2}{(z + \frac{1}{2} - \frac{j}{2})(z - \frac{1}{2} + \frac{j}{2})}$$

(b) Sketch the pole-zero diagram of the system.



(c) Roughly sketch the frequency response $|H(\Omega)|$ of the system. $\Omega = 0$; $z = e^{j0} = 1$



$$H(0) = \frac{1}{1 - 1 + 0.5} = 2$$

$\Omega = \pi$: $z = e^{j\pi} = -1$

$$H(\pi) = \frac{(-1)^2}{(-1)^2 - (-1) + 0.5} = 0.4$$

$\Omega = \frac{\pi}{4}$: $z = e^{j\pi/4}$ $\quad H(\frac{\pi}{4}) = 2.8$

(d) Find the output $y[n]$ of the system when the input is $x[n] = (0.5)^n u[n]$, with initial conditions $y[-1] = -1$, $y[-2] = -2$. Identify the zero-input and zero-state parts of the response.

$$Y(z) - 1(z^{-1}Y(z) + \overset{-1}{\cancel{y(-1)}}) + 0.5(z^{-2}Y(z) + z^{-1}\overset{-1}{\cancel{y(-1)}} + \overset{-2}{\cancel{y(-2)}}) = X(z)$$

$$Y(z)(1 - z^{-1} + 0.5z^{-2}) - 0.5z^{-1} = X(z)$$

$$Y(z)(z^2 - z + 0.5) = 0.5z + z^2X(z) = 0.5z + \frac{z^3}{z - 0.5}$$

$$Y(z) = \underbrace{\frac{0.5z}{z^2 - z + 0.5}}_{\substack{Y_{zi}(z) \\ \text{zero input}}} + \underbrace{\frac{z^3}{(z^2 - z + 0.5)(z - 0.5)}}_{\substack{Y_{zs}(z) \\ \text{zero state}}}$$

$$Y_{zi}(z) = \frac{-0.5z}{z^2 - z + 0.5}$$

$$\frac{Y_{zi}(z)}{z} = \frac{-0.5}{z(z^2 - z + 0.5)} = \frac{-j}{z - 0.5 + j0.5} + \frac{j}{z - 0.5 - j0.5}$$

$$Y_{zi}(z) = \frac{e^{-j\pi/2} z}{z - \frac{\sqrt{2}}{2} e^{j\pi/4}} + \frac{e^{j\pi/2} z}{z - \frac{\sqrt{2}}{2} e^{-j\pi/4}}$$

$$y_{zi}(n) = 2 \left(\frac{\sqrt{2}}{2}\right)^n \cos\left(\frac{\pi}{4} n - \frac{\pi}{2}\right) u(n)$$

$$Y_{sz}(z) = \frac{z^3}{(z^2 - z + 0.5)(z - 0.5)}$$

$$\frac{Y_{sz}(z)}{z} = \frac{z^2}{(z^2 - z + 0.5)(z - 0.5)} \quad \frac{-j}{z - 0.5 + j0.5} + \frac{j}{z - 0.5 - j0.5} + \frac{1}{z - 0.5}$$

$$Y_{zs}(z) = e^{-j\pi/2} \frac{z}{z - \frac{\sqrt{2}}{2} e^{j\frac{\pi}{4}}} + e^{j\frac{\pi}{2}} \frac{z}{z - \frac{\sqrt{2}}{2} e^{-j\pi/4}} + \frac{z}{z - 0.5}$$

$$y_{zs}(n) = 2 \left(\frac{\sqrt{2}}{2}\right)^n \cos\left(\frac{\pi}{4} n - \frac{\pi}{2}\right) u(n) + (0.5)^n u(n)$$