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1 TIME OF ARRIVAL LOCATION TECHNIQUE

1.1 TOA Principals

Introduction

Systems that measure the arrival times of signals can accurately locate the signals in 2- or 3-dimensional space and time. Among other things, this approach forms the basis for the Global Positioning System (GPS) that is providing revolutionary technological, societal, and scientific developments. Time-of-arrival (TOA) measurements, as they are called, are also used in locating cell phones, for seismological studies and in Tech's Lightning Mapping Array. In this laboratory we will develop the basic ideas behind TOA systems and explore the mathematics behind them.

TOA systems basically solve the equation velocity times time equals distance, $v \cdot t = d$, or more specifically, $v\delta t = \delta\ell$, where $\delta t = (t_i - t)$ is the difference between the arrival time t_i at location i and the source time t , and $\delta\ell$ is the distance between the the measurement location x_i, y_i, z_i and source location x, y, z . Thus, from the Pythagorean theorem, we have that

$$v(t_i - t) = \sqrt{(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2}. \quad (1)$$

Measuring t_i at 4 or more locations is sufficient to determine the 4 unknowns x , y , z , and t (Figure 1).

For simplicity we will consider the 2-dimensional case in which the source and measurement locations lie in the same z plane. For this case, $z_i = z$ and (1) becomes

$$v(t_i - t) = \sqrt{(x_i - x)^2 + (y_i - y)^2}. \quad (2)$$

The 3 unknowns x , y , t can be determined from measurements of t_i at 3 different locations.¹

The manner in which the arrival time measurements locate the source can be determined graphically from the fact that the *differences* in

¹This is not exactly true, either for the 2- or 3-dimensional case, as we shall see.

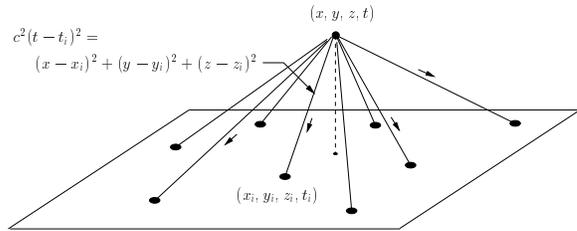


Figure 1: Geometry of TOA location technique.

the arrival times at a pair of stations i, j constrain the source to lie on a hyperboloid of revolution about the baseline between the two stations. This can be seen by reviewing the basic properties of ellipses and hyperbolas.

Ellipses and Hyperbolas

An ellipse has the property that sum of the distances from the two foci of the ellipse is a constant

$$d_1 + d_2 = 2a. \quad (3)$$

The equation for the ellipse shown in Figure 2 is

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1. \quad (4)$$

Exercise 1: Show that the x- and y- intercepts of the ellipse are at $x = \pm b$ and $y = \pm a$. Also show that the distance c of the foci from the x axis is related to a and b by

$$b^2 + c^2 = a^2. \quad (5)$$

(Hint: Look for a right triangle having the suggested sides.)

Hyperbolas, on the other hand, have the property that their locus of points has a constant *difference* from the two foci.

$$|d_1 - d_2| = 2a. \quad (6)$$

Since the difference can be either positive or negative, the property applies to the *magnitude* of the difference. Note that for a given set of foci, there are two hyperbolae that satisfy the

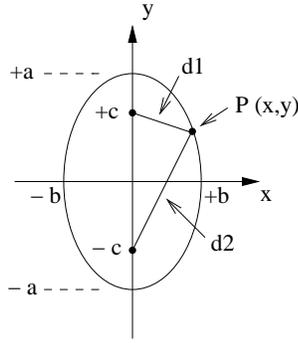


Figure 2: Geometry of an ellipse.

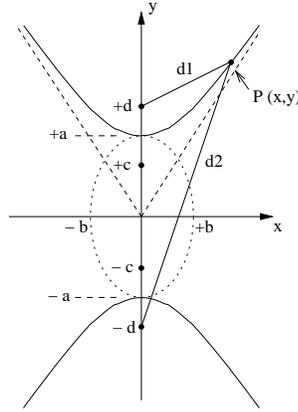


Figure 3: Geometry of hyperbolas.

relation (upward and downward), and that unlike the ellipses, they are open curves. The equation for the upward and downward hyperbolae shown in Figure 3 is the same as that for an ellipse, except for a minus sign in the second term:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \quad (7)$$

Whereas the foci of the equivalent ellipse are at $y = \pm c$, the foci of the hyperbolae are at $y = \pm d$. A similar relation exists between d and the parameters a and b , namely

$$a^2 + b^2 = d^2. \quad (8)$$

Exercise 2: Show that the y -intercepts of the hyperbolae are at $y = \pm a$, and that the asymptotes of the ellipse (the dashed lines in Figure 3 are given by

$$y = \pm \left(\frac{a}{b}\right) x. \quad (9)$$

Expression 8 relating a , b , and d is more difficult

to show – can you figure out how to derive it?

Hyperbolic TOA Relations

Considering the x,y locations of the two foci of the hyperbolae to correspond to the locations of two TOA measurement locations, it is not difficult to see that the locus of points of the hyperbolae, corresponding to a constant *distant* difference between the distances d_1 and d_2 , also corresponds to a constant or given *time* difference of arrival (TDOA) at the two locations. We will denote this time difference of arrival as Δt .

Exercise 3: Show that the above statement holds, and determine the parameters a and d for a pair of stations separated by a distance D (the ‘baseline’ length between the stations). [Partial answer: $a = v|\Delta t|/2$.]

Geometric Solutions:

Do the following:

- Write a .m program that plots the hyperbolae corresponding to $a = 5$ and $b = 3$, and the asymptotes of the hyperbolae. The plot should cover the range $x = \pm 30$ and $y = \pm 20$ units. Add the equivalent ellipse to the plot, as in Figure 3, and show the foci of both the hyperbolae and the ellipse. (An sample program is provided to get you started.) Make sure your program is commented to explain what you are doing. Save this program, and use it as a starting point for the other programs discussed below.

- To plot a rotated hyperbola, it is necessary to do a coordinate transformation. The transformation used to plot the hyperbola corresponding to a particular baseline takes points calculated in an x'', y'' coordinate system in which the baseline is vertically upward (in which the hyperbolae are easy to calculate), and rotates the points clockwise by an angle θ so that the y'' axis is correctly oriented with the actual baseline in the network’s x, y coordinate system. It then offsets the resulting points away from being centered around $(0,0)$ to the center (x_0, y_0) of the actual baseline. From a carefully drawn sketch of the geometry of the $('')$ to $(')$ transformation, determine the matrix expression for

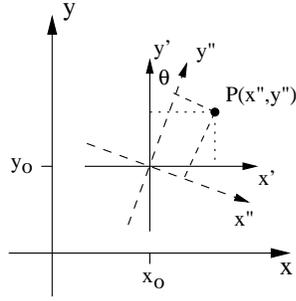


Figure 4: Coordinate transformation.

converting (x'', y'') values to (x', y') values, and confirm that it is the same as used in the transform program. You need to find the matrix coefficients so that:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x'' \\ y'' \end{bmatrix}$$

Do the same for the translation from the $(')$ to the unprimed system. Using these results, complete the attached Matlab function `xfm1`.

- Modify your hyperbola-plotting program to plot the hyperbolae when the center of the stations is at location $x_0 = 10$ and $y_0 = 5$ units, with the baseline between the stations rotated 60° from the vertical. Your plot should look similar to that shown in Figure 5. Use the completed function `xfm1` to rotate and translate the hyperbolae. Use the ‘axis equal’ command (do a ‘help axis’) to give a 1:1 aspect ratio so that the hyperbolae look correct.

- Again saving the program as a separate file, further modify it to show how the hyperbolae corresponding to TOA measurements at three stations locate a source in two spatial dimensions. Let the station locations be at the vertices of the isosceles triangle $x_1, y_1 = (-10, -10)$, $x_2, y_2 = (+10, -10)$, and $x_3, y_3 = (0, +10)$ km, respectively. Use Matlab’s ‘input’ function to specify a source location (also in km units), and calculate the times the signal would be received at each station. Finally, plot the hyperbolae corresponding to each of the three baselines to see the solution. Use the ‘pause’ function in your for loop to identify each hyperbola as it is added to the plot. Your plot should look something like Figure 6.

- An interesting property of the TOA solu-

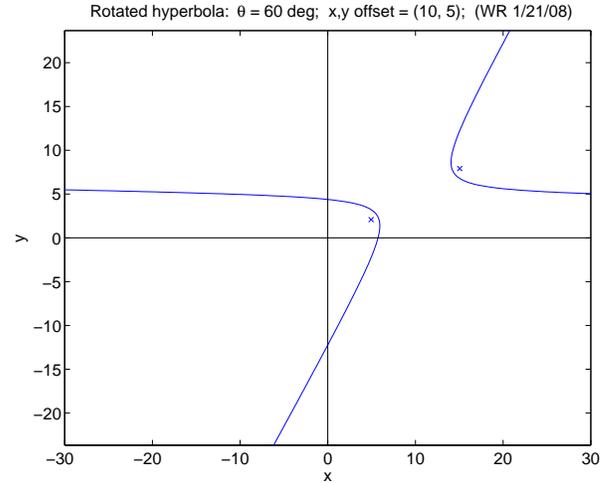


Figure 5: Rotated hyperbola.

tions is that they are not unique in some regions of the x, y domain when the number of measurements is just enough to determine the unknowns. For example, examine the hyperbolae for a source at $x, y = (1, 20)$ km. Use the manual zoom capabilities of the matlab plot window to confirm that there is a second solution. Where is the second solution located? Both locations have exactly the same time differences of arrival and therefore cannot be distinguished from 3-station measurements.

- Explore how ‘wide’ the multiple solution domain is in x when y remains fixed at 20 km. From the symmetry of the network, where else in the $x-y$ domain would you expect multiple solutions to exist? Confirm this and make a rough sketch of the multiple-solution regions on one of your hard copy plots.

- The solution ambiguity can be eliminated by adding a fourth station to the network. Again, keeping your 3-station program intact for comparison, do this to see if it is indeed true. (NOTE: Be sure that the first 3 baseline pairs remain the same when adding the 4th baseline.)

Analytic Solutions.

In the above, we solved the 2-dimensional time-of-arrival (TOA) equations graphically by plotting the hyperbolae corresponding to the differences in the arrival times at successive pairs (baselines) of measurement stations. In this man-

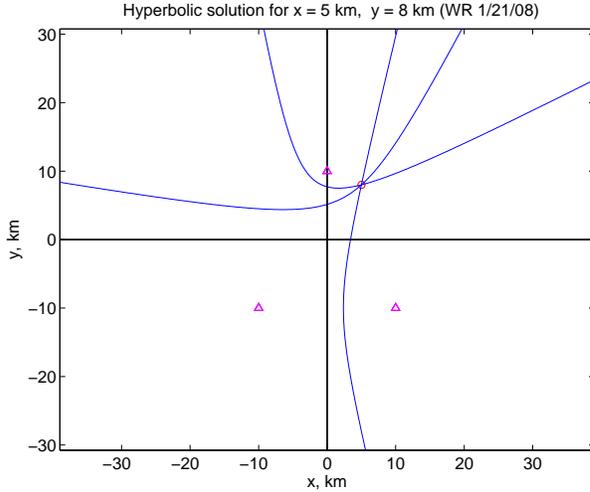


Figure 6: 3-station hyperbolic solution.

ner we could use arrival time measurements at 3 stations to determine the three unknowns (x, y, t) of the source location and time. We found that in some regions of the x, y plane the hyperbolae intersected at two locations, so that the solution was not unique. Multiple solutions occurred when the source was outside the network, in the radial ‘shadow’ of each station.

Although the predicted TOA values differed for the two possible source locations, the *differences* in the arrival times at pairs of stations turn out to be identical – hence the non-uniqueness. The solution could be made unique by adding a fourth station to generate an additional hyperbola that resolved the ambiguity. The need for an extra measurement derives fundamentally from the non-linear nature of the TOA equations, whose solutions can sometimes be multi-valued.

In the following, we show how measurements at four locations can be used to obtain an analytic solution to the problem, in the form of three linear equations in the three unknowns, that can be solved by standard linear algebra techniques. As before, the solution is unique.

Approach:

The basic way of solving a set of equations is by eliminating variables. From before, the general

equation for the 2-dimensional case is

$$c(t_i - t) = \sqrt{(x_i - x)^2 + (y_i - y)^2}, \quad (10)$$

where we now recognize that $v = c$, the speed of light in air.

The hyperbolic solution is obtained by eliminating t as variable from the equations for a set of two stations. Letting the 2nd station be station j , t can be eliminated by differencing the TOA equations for each station:

$$\begin{aligned} c(t_i - t) &= \sqrt{(x_i - x)^2 + (y_i - y)^2} \\ c(t_j - t) &= \sqrt{(x_j - x)^2 + (y_j - y)^2}, \end{aligned}$$

giving

$$\begin{aligned} c(t_i - t_j) &= \sqrt{(x_i - x)^2 + (y_i - y)^2} \\ &\quad - \sqrt{(x_j - x)^2 + (y_j - y)^2}. \end{aligned} \quad (11)$$

The result is a single equation in two unknowns (x, y) that is basically the equation for a hyperbola whose y -axis is the baseline. A second equation can be obtained from the difference of the third station measurement (station k) and either the i or j station, from which x, y can be determined. Given x and y , the source time t can then be determined from (10) for any one of the original measurements.

Another way of handling the equations is to eliminate the terms in x^2 , y^2 and t^2 , which linearizes things. This can be done by writing (10) in *squared* form,

$$c^2(t_i - t)^2 = (x_i - x)^2 + (y_i - y)^2, \quad (12)$$

and differencing them for pairs of stations, as above. The general result of differencing the i^{th} and j^{th} equations is

$$(x_i - x_j)x + (y_i - y_j)y - c(t_i - t_j)ct = k_{ij}$$

where $k_{ij} = (1/2)[(r_i^2 - r_j^2) - c^2(t_i^2 - t_j^2)]$, and $r_i^2 = x_i^2 + y_i^2$ etc. Differencing the equations for four stations gives three equations that are linear in the 3 unknowns, x, y, t . The equations

can be written in matrix form as

$$A\vec{x} = \vec{k}, \quad (13)$$

where A is the matrix of coefficients of the linear equations and $\vec{x} = x, y, t$ and \vec{k} are column vectors corresponding to the space-time source location and k_{ij} values, respectively.

- Derive the general linear equation and use the matrix capabilities of Matlab to implement the solution of (13) for a 4-station network. Add this to the 4-station version of your hyperbolic program from the previous set of exercises, and plot a cross-hair at the solution point in x, y space.

- Print out the solution values for x, y, t and evaluate the difference from the input values to see how well the two agree.

Starting programs

```
% Program to plot the hyperbola
%  $y^2/a^2 - x^2/b^2 = 1$ 
% The hyperbolae are open up/down, so that x is the independent variable
% for plotting. (Using the form  $x^2/a^2 - y^2/b^2 = 1$  requires that y be
% the independent variable, which is awkward programming-wise.)

clear % all variables
figure(1), hold off % start a new figure
set(gca,'FontSize',14) % adjust fontsize
xmax = 30; ymax = 20;
x = linspace(-xmax,xmax,1001); % array of x values for plot (why 1001?)
a = 5; b = 3;
y = ????????????; % corresponding y values
plot(x,y)
hold on % add to current plot
plot(x,-y) % Plot other half of hyperbola
axis([-xmax xmax -ymax ymax]) % specify axis limits
xlabel('x')
ylabel('y')
title(['Hyperbola  $y^2/a^2 - x^2/b^2 = 1$ ;  $a =$ ', num2str(a), ...
      ',  $b =$ ', num2str(b),']; (WR 1/21/08)'],'Interpreter','latex')
% Add axes
plot([0 0],[-ymax ymax],'k') % y axis (black line - 'k')
plot([-xmax xmax],[0 0],'k') % x axis
```

Transform subroutine

```
function [xout,yout] = xfm1(xin,yin,theta,x_offset,y_offset)
% Program to rotate and translate x,y values from x",y" to x,y space.
% Written to plot hyperbolas for time of arrival code.
% theta value assumed to be in radians.

% rotation matrix
xfm = ?????? ;
% make x,y values into a column vector
r_in = [xin; yin];
% rotate
r_out = xfm*r_in;
x = r_out(1,:); y = r_out(2,:);
xout = ?????;
yout = ?????;
```

1.2 Time of Arrival Location Error Analysis

Introduction

In the preceding section, it was assumed that there were no errors in the measurements. The locations are based on solving the Equation 1 for the three unknowns x, y, t based on the measured values x_i, y_i, t_i . The errors in x_i and y_i are usually small — with inexpensive GPS receivers, location errors are less than one meter, and with more expensive equipment, location errors are less than a millimeter. The largest source of error is in the measurements of the arrival times t_i . Using inexpensive GPS receivers, the errors in t_i are about 50 ns. In this section we will explore how much errors in the arrival time measurements affect the location accuracy of a TOA system.

Normally Distributed Errors

Things such as timing errors are usually have normal, or Gaussian, distributions. This means that the probability distribution can be written as:

$$f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2(x-m)^2} \quad (14)$$

Figure 7. shows a plot of $f(x)$ for $h = 1$ and $m = 5$.

The mean of a function is its average value, which can be calculated as:

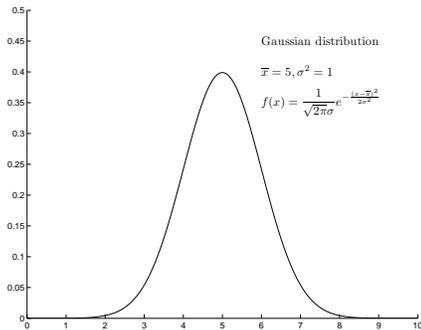


Figure 7: Gaussian probability distribution.

$$\bar{x} = \int_{-\infty}^{\infty} x f(x) dx \quad (15)$$

In Exercise 5, you will show that the mean of a normal distribution is $\bar{x} = m$, so:

$$f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2(x-\bar{x})^2} \quad (16)$$

The variance σ^2 of a function is the mean value of the square of the difference between the x and the mean value of x :

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x) dx \quad (17)$$

The variance shows how much of the distribution $f(x)$ is different than \bar{x} . The standard deviation is defined as the square root of the variance, so the standard deviation is σ . In Exercise 5 you will show that

$$\sigma = \frac{1}{\sqrt{2}h} \quad (18)$$

Thus, a Gaussian probability distribution can be written:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\bar{x})^2/2\sigma^2} \quad (19)$$

When a calculation depends on a number of variables which have errors, there is a way to calculate the error in the final value. If the output of a calculation is Q , and Q depends on measured values a and b , then it can be shown that

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial a} \right)^2 \sigma_a^2 + \left(\frac{\partial Q}{\partial b} \right)^2 \sigma_b^2 \quad (20)$$

Exercise 5:

1. Show that $\bar{x} = m$, and $\sigma^2 = 1/\sqrt{2}h$. (Hint: Integrate by using a change of variable $z = h(x-m)$. You then integrate by parts, or look up the resulting integrals in a table of integrals.)
2. Show that the standard deviation in the time difference of arrival $t_2 - t_1$ is

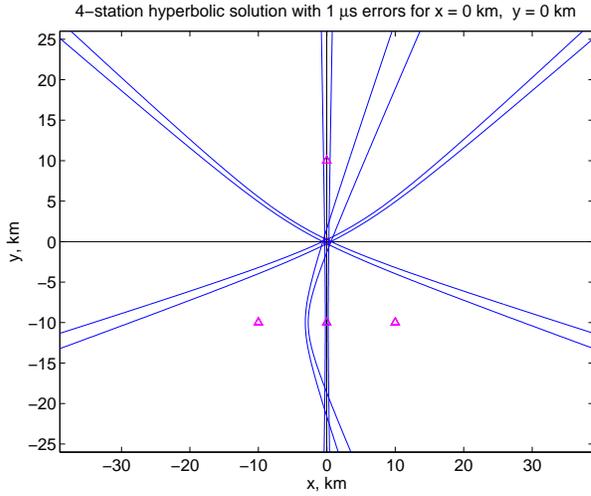


Figure 8: Error region inside of array.

$$\sigma_{t_1-t_2} = \sqrt{\sigma_{t_1}^2 + \sigma_{t_2}^2}$$

- Use the MATLAB function `normrnd` to generate a large number of random variables with a mean value of 5 and a standard deviation of 1. Use the `hist` function to show that the random variable distribution looks like that of Figure 7.

Graphical Esitimation of Errors

In Section 1, you plotted hyperbolae for a four-station TOA network. The point where the hyperbolae crossed was the location of the RF source. By modifying your program, it will be easy to see how large the location errors will be for a system due to the uncertainties in the measurements of the arrival times. Call the measurement uncertainty Δt . From Exercise 5, the uncertainty in the time difference is $\sqrt{2}\Delta t$. If you replot the hyperbolae when the time-of-arrival differences are this much smaller and this much larger than the actual arrival times, you will get a plot which looks like Figure 8 (using $\Delta t = 1\mu s$):

If you blow up the region containing the source, the plot look like Figure 9.

About 68% of the source locations should lie inside the region which is common to all the

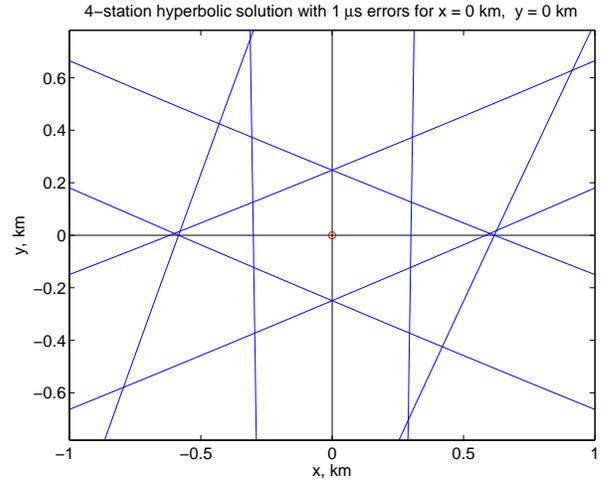


Figure 9: Enlarged view of region of Figure 8. .

hyperbolae. Note that this region is about as wide as it is long.

The above figures were generated for sources inside the network. When the source is outside the netork, the common region of the hyperbolae look like Figure 10

The common region is much longer than it is wide. The reason for this will be discussed later.

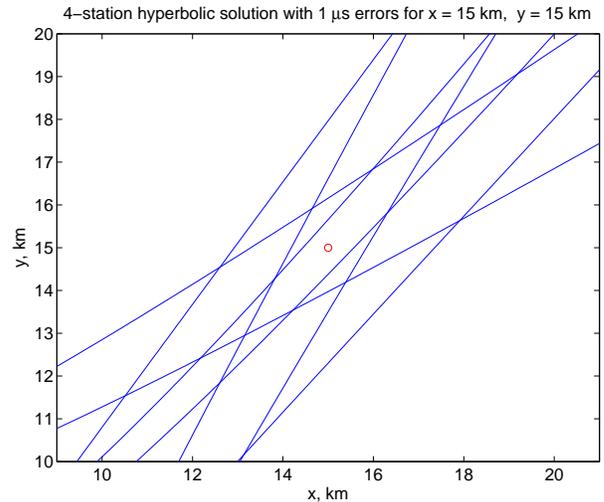


Figure 10: Error region outside of array.

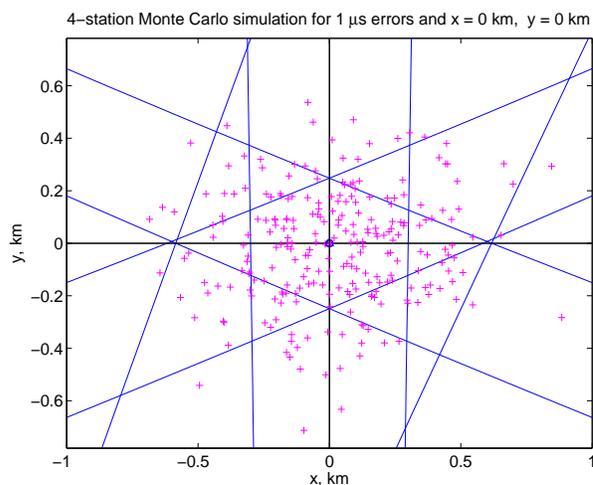


Figure 11: Monte Carlo simulation, 250 points.

Monte Carlo Error Analysis

The above graphical analysis shows where you would expect a real system to calculate a source location compared to the actual source location. To check out whether this is what will happen, you can use a technique called Monte Carlo analysis. To perform a Monte Carlo analysis, you start with a source location, just as you did before. You then add random timing errors (using a computer program's random value function) to the actual time-of-arrival values, and solve for the source location using the four-station analytical method, using the modified arrival times. You do this for a large number of different samples (using different timing errors). Figure 11 shows a Monte Carlo simulation of the toa-determined location of a source at the origin of the array, for a system with $1 \mu\text{s}$ errors.

Note that most, but not all, samples lie inside the area common to all hyperbolae. For a Gaussian distribution, about 68% of the samples will lie within one standard deviation, so about 68% of the points should lie within the common area.

Exercise 6: Modify your programs from last week to generate plots similar to those of Figures 8, 9, 10, and 11, for a time-of-arrival network with a timing uncertainty of 100 ns. Also, do a Monte Carlo simulation for a source outside of the array. In the Monte Carlo simula-

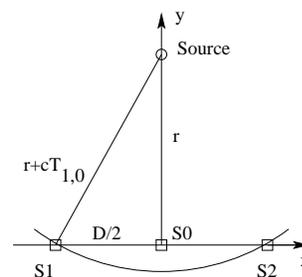


Figure 12: Simple model for range determination.

tions, verify that about 68% of the simulated source locations lie inside the common area of the ellipse.

Simple Model for Errors Outside of Array

When a source is inside of the array, the location errors tend to be about the same size as the timing errors (when the timing errors are converted to distance). This is because, as a source moves away from one station, it moves closer to another, so a Δd position change results in a $2\Delta d/c$ time-of-arrival distance. When a source is outside the array, as it moves away from one station, it moves away from all stations, so the time-of-arrival differences are much smaller than when a source inside. The effect can be illustrated with a simple model. Figure 12 shows how the range to a source (distance from the array to the source) is determined when the source is outside the array. The range is determined by the difference in time between the signal reaching Station 0 and the signal reaching Station 1.

Figure 13 shows how the azimuth to a source is determined when the source is outside the array. The azimuth is determined by the difference in time between the signal reaching Station 2 and the signal reaching Station 1.

Exercise 7: Using the simple model above, show that range is

$$r \simeq \frac{D^2}{8cT_{1,0}}$$

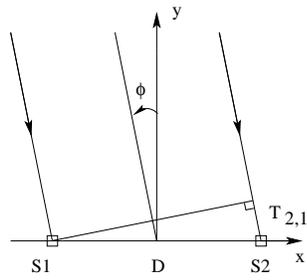


Figure 13: Simple model for azimuth determination.

and that the uncertainty in range is

$$\Delta r \simeq 8 \left(\frac{r}{D} \right)^2 c \Delta T_{1,0}$$

Show that, for small ϕ , the error in azimuth is

$$\Delta \phi = \left(\frac{1}{D} \right) c \Delta T_{2,1}$$

and the resulting uncertainty in the x direction is

$$\Delta y = r \Delta \phi = \left(\frac{r}{D} \right) c \Delta T_{2,1}$$

1.3 Determining the Best Location for Multi-Station Time of Arrival Measurements

Introduction

With a four-station network, one can find a location from the three linear equations for time differences (Equation 11). If there are no uncertainties in the measurements, the location will be exact, and will be the solution to the non-linear equations

$$t_i = t + \frac{\sqrt{(x_i - x)^2 + (y_i - y)^2}}{c} \quad (21)$$

If, however, there are uncertainties in the measurements, the solution will fit the linear equations for time differences, but will not exactly fit the non-linear equations. For example, suppose we have the following station locations and arrival times, where the arrival times have 40 ns uncertainties:

| Station | x | y | t_i |
|---------|-----|-----|-----------------------|
| 1 | 0 | 0 | 0.0000 μs |
| 2 | 0 | 10 | 25.7830 μs |
| 3 | -10 | -10 | 37.0099 μs |
| 4 | -10 | 10 | 9.2517 μs |

The linear time difference of arrival equations give a location of $x = 5.3936$ km, $y = -2.5713$ km, and $t = -19.8973\mu\text{s}$. However, if we put these values of x , y and t into Equation 21, the calculated arrival times differ slightly from the actual arrival times:

| Station | t_i (measured) | t_i (calculated) |
|---------|-----------------------|-----------------------|
| 1 | 0.0000 μs | 0.0335 μs |
| 2 | 25.7830 μs | 25.7976 μs |
| 3 | 37.0099 μs | 37.0216 μs |
| 4 | 9.2517 μs | 9.2745 μs |

If there were more than four stations, how could you find the position of a source? You could find locations by taking different combinations of three time differences of arrival (all

of which would give slightly different answers). For example, with a five-station network, you could use the three time differences of $(t_2 - t_1)$, $(t_3 - t_1)$ and $(t_4 - t_1)$ to find one location, then use the three time differences of $(t_2 - t_1)$, $(t_3 - t_1)$ and $(t_5 - t_1)$ to get another location. Other combinations would give yet other locations, which could be averaged.

There is a better way to get a location. If there were no errors in the measurements, the equation

$$ct_i - ct - \sqrt{(x_i - x)^2 + (y_i - y)^2} \quad (22)$$

will equal zero for every station i . If there are errors in the measurement of t_i , then Equation 23 will be the error of the time measurement at station i , which will be about the standard deviation σ_{ct} of the error of the time measurement. If we take the square of Equation 22, then the value will always be positive, and will be about the value of the variance σ_{ct}^2 of the error in the time measurement:

$$\left(ct_i - ct - \sqrt{(x_i - x)^2 + (y_i - y)^2}\right)^2 \quad (23)$$

If there are errors in the time measurement, then it will be impossible to determine the true source location and time. However, a “good” location is one where the value of Equation 23 is small for every station i . We define the quantity χ^2 as:

$$\chi^2 = \sum_{i=1}^N \frac{\left(ct_i - ct - \sqrt{(x_i - x)^2 + (y_i - y)^2}\right)^2}{\sigma_{ct}^2} \quad (24)$$

then the “best” location will be the one which makes χ^2 the smallest. Because the values in Equation 23 will be about the size of σ_{ct}^2 , χ^2 will be on the order of unity for the “best” fit.

In a course in statistics, it can be shown the quantity χ_ν^2 (called the reduced χ^2) has a value of about 1 for the “best” fit:

$$\chi_\nu^2 = \sum_{i=1}^N \frac{(ct_i - ct - \sqrt{(x_i - x)^2 + (y_i - y)^2})^2}{(N - 3)\sigma_{ct}^2} \quad (25)$$

(The $(N - 3)$ on the bottom is the number of “degrees of freedom” in the measurement. We need to solve for three quantities. so we need at least three measurements. If we have a five station network, then there are $5 - 3 = 2$ extra, or redundant, measurements. It is said that there are two degrees of freedom when we have five measurements to solve for three unknowns.)

How can we use χ_ν^2 to find the “best” values of x , y , and ct ? Let’s take an example of a system where we are trying to find one value rather than three. For example, suppose you are asked to find the coefficient of thermal expansion of a metal. You take a bar of metal 1 meter long, and measure its length as a function of temperature. The equation for length is $L = L_0(1 + \alpha(T - T_0))$, where L_0 is the length at a standard temperature T_0 , T is the temperature at which the measurement is made, and α is the coefficient of thermal expansion. Then the reduced χ_ν^2 for this experiment is:

$$\chi_\nu^2 = \sum_{i=1}^N \frac{(L_i - L_0(1 - \alpha(T - T_i)))^2}{(N - 1)\sigma^2} \quad (26)$$

where L_i is the length of measurement number i , T_i is the temperature of measurement i , and σ^2 is the variance in the measurement. (σ for the numerator can be calculated using Equation 20.) The “best” α will be the one which makes χ_ν^2 smallest. If we plot χ_ν^2 vs. α , we might get a plot which looks something like Figure 14. To find the “best” α , we start with a good guess (say, $\alpha = 20 \times 10^{-6} \text{ K}^{-1}$, a value typical for a metal), and then we can develop a search method to find the value of α which makes χ_ν^2 smallest. For the example of Figure 14, χ_ν^2 will be 0.972 for the “best” α of $17.3 \times 10^{-6} \text{ K}^{-1}$

Suppose you had a series of measurements where you were asked to find two parameters, α

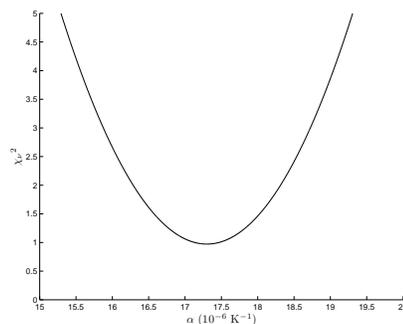


Figure 14: Plot of χ_ν^2 vs. α .

and β . Then the reduced χ^2 will depend on both α and β , and a plot of χ_ν^2 vs. α and β might look something like Figure 15. A search over both α and β to locate the smallest value of χ_ν^2 will find the “best” values for these parameters.

Let’s go back to the case of a time-of-arrival system. We want to find the “best” value for the three parameters, x , y and ct from N time-of-arrival measurements. Suppose we have a six-station network. We can get an initial guess for the parameters by solving the linear equation for any three time differences of arrival. We can find the best fit by searching around this initial guess for the values of x , y and ct which give the lowest value of χ_ν^2 . Here is a simple (but inefficient) algorithm: make a grid around the initial guess (for x , y and ct) which goes one kilometer in each direction with 100 meter steps. Find the location in this grid which gives the smallest χ_ν^2 . Using this as a new guess, search over a new grid which is 100 m x 100 m

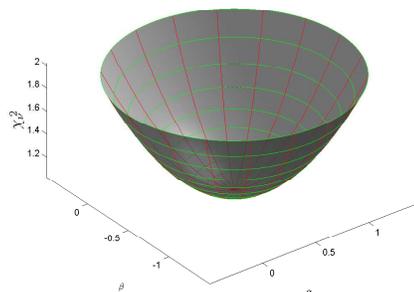


Figure 15: Plot of χ_ν^2 vs. α .

x 100 m, with steps of 10 m. From the best value in this grid, create a new 10 m x 10 m x 10 m grid with steps of 1 m. You can use the value in this grid which gives the lowest χ_ν^2 as your “best” value. There is no need to take any smaller grid, because with 40 ns timing errors, we know that the closest we could possibly get to the actual location is about 15 m.

In class, we will discuss more efficient algorithms for finding the “best” values for x , y and ct .

Exercise 8:

1. Download the files `stations.loc` and `toa_data.dat`. The file `stations.loc` has the x - y locations of eight TOA stations. (The first column is the station number, the second column is the x position in kilometers, and the third column is the y position in kilometers.) The file `toa_data.dat` has time-of-arrival data for 1,000 sources. The file consists of eight columns, with the first column being the time (in seconds) the signal arrives at Station 1, etc.
2. Write a program to generate linear solutions to the TOA data. Pick one station as a reference station, and use any three other stations to get three time differences of arrival. Generate a file `linear_solution.dat` which has the x , y and t values from the linear solutions.
3. Write a program to find the “best” fit to the data; i.e. the locations which give the smallest χ_ν^2 . Generate a file `least_squares_solution.dat` which has the x , y , t and χ_ν^2 values from the least squares solution.
4. Use MATLAB to plot the x and y locations of the least-squares solutions.
5. The points should lie in an obvious pattern. Comment on the size of the deviation of the pattern from a smooth curve as a function of the distance from the center of the array.
6. Use MATLAB to plot a histogram of the number of solutions vs. the value of χ_ν^2 .