TOA Equation

The TOA equation is:

$$c(t_i - t_j) = \sqrt{(x - x_i)^2 + (y - y_i)^2} - \sqrt{(x - x_j)^2 + (y - y_j)^2}$$

Looking at Figure 3, we have $y_i = d$, $y_j = -d$, and $x_i = x_j = 0$. Then

$$c(t_i - t_j) = \sqrt{x^2 + (y - d)^2} - \sqrt{x^2 + (y + d)^2}$$

Let $c\Delta t_{ij} = c(t_i - t_j)$, $d_i = \sqrt{x^2 + (y - d)^2}$ and $d_j = \sqrt{x^2 + (y + d)^2}$. Then:

$$\begin{aligned}
c\Delta t_{ij} &= d_i - dj \\
(c\Delta t_{ij})^2 &= (d_i - dj)^2 \\
(c\Delta t_{ij})^2 &= d_i^2 - 2d_i d_j + d_j^2 \\
-2d_i d_j &= (c\Delta t_{ij})^2 - d_i^2 - d_j^2 \\
(-2d_i d_j)^2 &= ((c\Delta t_{ij})^2 - d_i^2 - d_j^2)^2 \\
4d_i^2 d_j^2 &= (c\Delta t_{ij})^4 + d_i^4 + d_j^4 - 2(c\Delta t_{ij})^2 d_i^2 - 2(c\Delta t_{ij})^2 d_j^2 + 2d_i^2 d_j^2 \\
0 &= (c\Delta t_{ij})^4 + d_i^4 + d_j^4 - 2(c\Delta t_{ij})^2 d_i^2 - 2(c\Delta t_{ij})^2 d_j^2 - 2d_i^2 d_j^2 \\
0 &= (c\Delta t_{ij})^4 + (d_i^4 - 2d_i^2 d_j^2 + d_j^4) - 2(c\Delta t_{ij})^2 d_i^2 - 2(c\Delta t_{ij})^2 d_j^2 \\
0 &= (c\Delta t_{ij})^4 + (d_i^2 - d_j^2)^2 - 2(c\Delta t_{ij})^2 (d_i^2 + d_j^2)
\end{aligned}$$

Now substitute in for d_i and d_j, and all the square roots go away:

$$\begin{aligned}
d_i^2 + d_j^2 &= 2x^2 + 2y^2 + 2d^2 \\
d_i^2 - d_j^2 &= -4dy
\end{aligned}$$

$$\begin{aligned}
0 &= (c\Delta t_{ij})^4 + (d_i^2 - d_j^2)^2 - 2(c\Delta t_{ij})^2 (d_i^2 + d_j^2) \\
0 &= (c\Delta t_{ij})^4 + (-4dy)^2 - 2(c\Delta t_{ij})^2 (2x^2 + 2y^2 + 2d^2) \\
0 &= (c\Delta t_{ij})^4 + 16d^2 y^2 - 4(c\Delta t_{ij})^2 x^2 - 4(c\Delta t_{ij})^2 y^2 - 4(c\Delta t_{ij})^2 d^2 \\
0 &= (c\Delta t_{ij}/2)^4 + d^2 y^2 - (c\Delta t_{ij}/2)^2 x^2 - (c\Delta t_{ij}/2)^2 y^2 - (c\Delta t_{ij}/2)^2 d^2 \\
(c\Delta t_{ij}/2)^2 y^2 - d^2 y^2 + (c\Delta t_{ij}/2)^2 x^2 &= (c\Delta t_{ij}/2)^4 - (c\Delta t_{ij}/2)^2 d^2 \\
((c\Delta t_{ij}/2)^2 - d^2) y^2 + (c\Delta t_{ij}/2)^2 x^2 &= (c\Delta t_{ij}/2)^2 ((c\Delta t_{ij}/2)^2 - d^2)
\end{aligned}$$

$$\frac{y^2}{(c\Delta t_{ij}/2)^2} + \frac{x^2}{((c\Delta t_{ij}/2)^2 - d^2)} = 1 \tag{1}$$

This is the equation for an ellipse (if $|c\Delta t_{ij}/2| > d$) or a hyperbola (if $|c\Delta t_{ij}/2| < d$). For a TOA problem, we must have $|c\Delta t_{ij}/2| < d$, (why?) so this is the equation of a hyperbola. Let $(c\Delta t_{ij}/2)^2 = a^2$ and $(d^2 - (c\Delta t_{ij}/2)^2) = b^2$. Then we have:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

the equation of a hyperbola.