## EE 451

### Solution of Difference Equations in the Time Domain

The difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

(where $a_o = 1$) has the solution

$$y[n] = \underbrace{\{C_1\lambda_1^n + \cdots + C_N\lambda_N^n\}\,u[n] + D_0\delta[n] + \cdots + D_{M-N-1}\delta[n-(M-N-1)]}_{y_{tr}[n]} + \underbrace{y_p[n]}_{y_{ss}[n]} \qquad (1)$$

where the $\lambda_k$'s are the roots of the characteristic polynomaial of the system:

$$\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N = 0.$$

(For repeated roots, use $\lambda_k^n$, $n\lambda_k^n$, $n^2\lambda_k^n$, etc. For example, if $\lambda_1 = \lambda_2$, you would use $C_1\lambda_1^n + C_2 n\lambda_1^n$ instead of $C_1\lambda_1^n + C_2\lambda_1^n$.)

$y_p[n]$ is the particular (steay-state) solution which depends on the input:

| $x[n]$ | $y_p[n]$ |
|---|---|
| $Au[n]$ | $Ku[n]$ |
| $A\alpha^n u[n]$ | $K\alpha^n u[n]$ |
| $A\cos[\omega_o n] + B\sin[\omega_o n]$ | $K_1\cos[\omega_o n] + K_2\sin[\omega_o n]$ |

If the characteristic polynomial has a root at the value of an input exponential (e.g., $\lambda_1 = \frac{1}{2}$ and $x[n] = \left(\frac{1}{2}\right)^n$) you would use $Kn\lambda_k^n$ for $y_p[n]$ (e.g., $y_p[n] = Kn\left(\frac{1}{2}\right)^n$).

There are $M - N$ $D_i$'s. If $M \le N$ there are no $D_i$'s.

To find the impulse response, there is no $y_p[n]$, and there are $M - N + 1$ $D_i$'s. If $M < N$ there are no $D_i$'s for the impulse response.

Solve for the unknowns by finding $y[0]$, $y[1]$, $y[2]$, $\cdots$ until you get as many equations as you have unknowns. Solve these equations for the unknowns.

You can solve the difference equation for the case $x[n] = 0$, subject to the initial conditions of the system. This is the *natural response* or *zero-input response*, $y_{zi}[n]$, of the system. You can further solve for the case $y[n] = 0$ for $n < 0$ for the actual input $x[n]$. This is the *forced response* or *zero-state response*, $y_{zs}[n]$, of the system. The total response is the sum of the two: $y[n] = y_{zi}[n] + y_{zs}[n]$.

A system is *BIBO stable* if, for every bounded input, the output is bounded. If the input $x[n]$ is bounded, then the particular solution $y_p[n]$ will be bounded. Thus, the only possible unbounded terms in Eq. (1) are the $\lambda_k^n$'s. These terms are bounded ($\lambda_k^n \to 0$ as $n \to \infty$) if $|\lambda_k| < 1$. Hence, the system is BIBO stable if $|\lambda_k| < 1$ for all the $k$'s. ($\lambda_k^n$ doesn't blow up if $|\lambda_k| = 1$. However, if $|\lambda_k| = 1$, the input $x[n] = \lambda_k^n$ will produce the output $y[n] = C_1\lambda_k^n + Kn\lambda_k^n$, and the $Kn\lambda_k^n$ term will blow up as $n \to \infty$.)