EE 451

Solution of Difference Equations in the Time Domain

The difference equation

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

(where $a_o = 1$) has the solution

$$y(n) = \underbrace{C_1 p_1^n + \cdots + C_N p_N^n + K_1 \delta(n) + \cdots + K_{M-N-1}\delta(n-(M-N-1))}_{y_{tr}(n)} + \underbrace{y_p(n)}_{y_{ss}(n)} \tag{1}$$

where the p_k's are the roots of the characteristic polynomaial of the system:

$$p^N + a_1 p^{N-1} + \cdots + a_{N-1}p + a_N = 0.$$

(For repeated roots, use p_k^n, np_k^n, $n^2 p_k^n$, etc. For example, if $p_1 = p_2$, you would use $C_1 p_1^n + C_2 n p_1^n$ instead of $C_1 p_1^n + C_2 p_1^n$.)

$y_p(n)$ is the particular (steay-state) solution which depends on the input:

$x(n)$	$y_p(n)$
$Au(n)$	$Du(n)$
$A\alpha^n u(n)$	$D\alpha^n u(n)$
$A\cos(\omega_o n) + B\sin(\omega_o n)$	$D_1\cos(\omega_o n) + D_2 \sin(\omega_o n)$

If the characteristic polynomial has a root at the value of an input exponential (e.g., $p_1 = \frac{1}{2}$ and $x(n) = \left(\frac{1}{2}\right)^n$) you would use Dnp_k^n for $y_p(n)$ (e.g., $y_p(n) = Dn\left(\frac{1}{2}\right)^n$).

There are $M-N$ K_i's. If $M \leq N$ there are no K_i's.

To find the impulse response, there is no $y_p(n)$, and there are $M-N+1$ K_i's. If $M < N$ there are no K_i's for the impulse response.

Solve for the unknowns by finding $y(0)$, $y(1)$, $y(2)$, $\cdots$ until you get as many equations as you have unknowns. Solve these equations for the unknowns.

To find the difference equation from a known response (e.g., the impulse or step response) do the following:

1. The p_i's are the roots of the characteristic polynomial. N is the number of p_i's. Find the characteristic polynomial:

$$p^N + a_1 p^{N-1} + \cdots + a_{N-1}p + a_N = (p-p_1)(p-p_2)\cdots(p-p_N).$$

This will give you the a_i's.

2. Determine the order M of the input side from the largest delay of the $\delta(n-d)'s$. If there is no $\delta(n-d)$, assume $M = N$ (assume $M = N-1$ for the impulse response). If there are $\delta(n-d)$'s, then $M = N + D + 1$, where D is the largest delayed $\delta(n-d)$. ($M = N + D$ for the impulse response).

3. Find $y(0)$, $y(1)$, $\cdots$, $y(M)$ from the equation for the response. Solve for b_0, b_1, ..., b_M.

The *transient response* of the system is $y_{tr} = C_1 p_1^n + \cdots + C_N p_N^n + K_1 \delta(n) + \cdots + K_{M-N} \delta(n - (M-N))$ – it is the part of the response which dies out in time. The *steady-state response* of the system is $y_{ss}(n) = y_p(n)$ – it stays around as long as the input drives it. The total response is the sum of the two: $y(n) = y_{tr}(n) + y_{ss}(n)$.

You can solve the difference equation for the case $x(n) = 0$, subject to the initial conditions of the system. This is the *natural response* or *zero-input response*, $y_{zi}(n)$, of the system. You can further solve for the case $y(n) = 0$ for $n < 0$ for the actual input $x(n)$. This is the *forced response* or *zero-state response*, $y_{zs}(n)$, of the system. The total response is the sum of the two: $y(n) = y_{zi}(n) + y_{zs}(n)$.

A system is *BIBO stable* if, for every bounded input, the output is bounded. The only possible unbounded terms in Eqn. (1) are the p_k^n's. These terms are bounded ($p_k^n \to 0$ as $n \to \infty$) if $|p_k^n| < 1$. Hence, the system is BIBO stable if $|p_k| < 1$ for all the k's. (p_k^n doesn't blow up if $|p_k| = 1$. However, if $|p_k| = 1$, the input $x(n) = p_k^n$ will produce the output $y(n) = C_1 p_k^n + Dnp_k^n$, and the Dnp_k^n term will blow up as $n \to \infty$.)