1.0 ANGULAR & LINEAR VELOCITY

1.1 DIFFERATION OF A ROTATION MATRIX:

\[ \dot{C}^a_b(t) = \Omega^{ab} C^a_b = C^a_b \Omega^{ab} \]

1.2 PROPERTIES OF SKEW SYMMETRIC MATRICES:

Multiplication by the skew-symmetric matrix is the same as the cross product of the vector defining the skew-symmetric matrix (\( \mathcal{K} = Sk\left(\hat{k}\right) = \left[ \hat{k} \times \right] \))

\[ \mathcal{K} \ddot{a} = Sk\left(\hat{k}\right) \ddot{a} = \left[ \begin{array}{ccc} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{array} \right] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \left[ k_3 a_1 - k_2 a_2 \\ k_2 a_1 - k_1 a_3 \\ k_1 a_2 - k_3 a_1 \right] = \hat{k} \times \ddot{a} \] (1)

Relationship betw. Rotation and skew-symmetric matrix:

\[ C_{(\hat{k}, \hat{\omega}(t))} = e^{\mathcal{K} \hat{\omega}(t)} = I + \sin(\hat{\theta}(t))\mathcal{K} + \left[1 - \cos(\hat{\theta}(t))\right] \mathcal{K}^2 \] (2)

Rotating skew-symmetric matrices:

\[ C_{Sk\left(\hat{\omega}\right)} C^T = Sk\left(C\hat{\omega}\right) \] (3)

or

\[ C_{Sk\left(\hat{\omega}\right)} = Sk\left(C\hat{\omega}\right) C \] (4)

1.3 A MOVING POINT (ORIGIN OF FRAME \{2\}) IN A ROTATING IN A ROTATING FRAME \{1\} AS SEEN FROM AN INERTIAL FRAME \{0\}:

- The position of frame 2 wrt frame 0 remains:

\[ \ddot{r}^{0}_{02}(t) = C^0_1(t) \ddot{r}^{1}_{12}(t) \] (5)

- The velocity of frame 2 wrt frame 0 becomes:

\[ \ddot{r}^{0}_{02} = \ddot{\omega}^{0}_{01}(t) \times \dot{r}^{0}_{12}(t) + C^0_1(t) \dot{r}^{1}_{12}(t) \] (6)

- The acceleration of frame 2 wrt frame 0 becomes:

\[ \dddot{r}^{0}_{02} = \dddot{\omega}^{0}_{01}(t) \times \ddot{r}^{0}_{12}(t) + \ddot{\omega}^{0}_{01}(t) \times \dot{\dddot{r}^{0}_{12}(t)} + 2 \dot{\omega}^{0}_{01}(t) \times \dot{\dddot{r}^{0}_{12}(t)} + \left( C^0_1 \dddot{r}^{1}_{12}(t) + C^0_1 \dddot{r}^{1}_{12} \right) \] (7)

Relationship betw specific force and acceleration (\( \dddot{g}^c_b = \dddot{r}^{c}_{ib} - \Omega^{c}_{ic} \dot{\Omega}^{c}_{ic} \dddot{r}^{c}_{eb} \)):

\[ \dddot{f}^{c}_{ib} = \dddot{a}^{c}_{ib} - \dddot{r}^{c}_{ib} \]


2.0 NAVIGATION RELATED COORDINATE SYSTEMS

2.1 THE EARTH-CENTERED EARTH-FIXED (ECEF) FRAME

The Earth rate vector described in the ECI frame becomes

\[
\vec{\omega}_e = \begin{bmatrix}
0 \\
0 \\
\omega_e
\end{bmatrix} = \vec{\omega}_e
\]

The orientation of the ECEF frame relative the ECI frame is \( \theta_e = \theta_{GMST} + \omega_nt \):

\[
C_e^e = R_{(\zeta, \delta)} = \begin{bmatrix}
\cos(\theta_e) & -\sin(\theta_e) & 0 \\
\sin(\theta_e) & \cos(\theta_e) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The position of the origin of the ECEF frame as seen in the ECI frame is zero, thus

\[
\vec{r}_e^i = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = \vec{r}_e = \vec{r}_e
\]

2.2 THE NAVIGATION (LOCALLY LEVEL) FRAME

The orientation of the Navigation frame \( wrt \) the ECEF frame is:

\[
C_n^e = R_{(\zeta, \delta)} R_{(\gamma, -L_n, -90')} = \begin{bmatrix}
-\cos(\lambda_p)\sin(L_p) & -\sin(\lambda_p) & -\cos(L_p)\cos(\lambda_p) \\
-\sin(L_p)\sin(\lambda_p) & \cos(\lambda_p) & -\cos(L_p)\sin(\lambda_p) \\
\cos(L_p) & 0 & -\sin(L_p)
\end{bmatrix}
\]

The angular velocity of the navigation frame \( wrt \) the ECEF frame is \( \dot{C}_n(t) = \Omega^e_n C_n^e \):

\[
\vec{\omega}_e^i = \begin{bmatrix}
\sin(\lambda_p)\dot{L}_p \\
-\cos(\lambda_p)\dot{L}_p \\
\dot{\lambda}_p
\end{bmatrix}
\]

The angular velocity of the navigation frame \( wrt \) the inertial frame is:

\[
\vec{\omega}_i^i = \vec{\omega}_e^i + C_e^e \vec{\omega}_e^i
\]

The position of the origin of the navigation frame \( wrt \) the ECEF frame in the ECEF frame is:
\[
\begin{bmatrix}
\cos(L_b)\cos(\lambda_b)(R_E + h_b) \\
\cos(L_b)\sin(\lambda_b)(R_E + h_b) \\
\sin(L_b)(1 - e^2)R_E + h_b
\end{bmatrix}
\]

After much effort (courtesy of Mathematica) it can be shown that:

\[
\vec{v}_{eb}^n = \begin{pmatrix}
\frac{(R_N + h_b)\hat{L}_b}{\cos(L_b)(R_E + h_b)} \\
-\hat{h}_b \\
\end{pmatrix}
\]

and hence, the time derivative of the curvilinear coordinates are readily obtained as:

\[
\begin{pmatrix}
\dot{L}_b \\
\dot{\lambda}_b \\
\dot{h}_b
\end{pmatrix} = \begin{pmatrix}
\frac{\vec{v}_{eb,N}^n}{(R_N + h_b)} \\
\frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E + h_b)} \\
\vec{v}_{eb,D}^n
\end{pmatrix}
\]

2.3 THE BODY FRAME

The orientation of the Body frame wrt the Navigation frame first Yaw (\(\psi\)) about the \(n\)-frame \(z\)-axis, then Pitch (\(\theta\)) about the new \(y\)-axis, and finally Roll (\(\alpha\)) about the new \(x\)-axis:

\[
C_b^n = R_{\hat{z}}(\psi)R_{\hat{y}}(\theta)R_{\hat{x}}(\phi)
\]

\[
= \begin{pmatrix}
c_\psi c_\phi & c_\psi s_\phi s_\theta - c_\theta s_\psi & c_\phi s_\psi + c_\theta s_\psi \\
c_\psi s_\phi & c_\psi c_\phi s_\theta + s_\psi s_\theta & c_\phi c_\psi s_\theta - s_\psi s_\theta \\
-s_\phi & c_\phi s_\phi & c_\phi c_\phi
\end{pmatrix}
\]

The angular velocity of the body frame wrt the inertial frame resolved in the inertial frame is

\[
\vec{\omega}_b^i = \vec{\omega}_m^i + C_{\psi}^i \vec{\omega}_{eb}^n
\]

The position of the origin of the body frame is the same as that of the nav frame:

\[
\vec{r}_{eb}^e = \vec{r}_{en}^e = \vec{r}_{in}^e = \vec{r}_{ib}^e
\]

since the origins of the ECEF and ECI frames are also coincident.