EE 451

Solution of Difference Equations in the Time Domain

The difference equation
\[ \sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k] \]
(where \( a_0 = 1 \)) has the solution
\[ y[n] = \left\{ C_1 \lambda_1^n + \cdots + C_N \lambda_N^n \right\} u[n] + D_0 \delta[n] + \cdots + D_{M-N-1} \delta[n - (M - N - 1)] + y_p[n] \]
\[ y_{\infty}[n] \]
(1)

where the \( \lambda_k \)'s are the roots of the characteristic polynomial of the system:
\[ \lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N = 0. \]

(For repeated roots, use \( \lambda_k^n \), \( n \lambda_k^n \), \( n^2 \lambda_k^n \), etc. For example, if \( \lambda_1 = \lambda_2 \), you would use \( C_1 \lambda_1^n + C_2 n \lambda_1^n \) instead of \( C_1 \lambda_1^n + C_2 \lambda_1^n \).

\( y_p[n] \) is the particular (steady-state) solution which depends on the input:

<table>
<thead>
<tr>
<th>( x[n] )</th>
<th>( y_p[n] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A u[n] )</td>
<td>( K u[n] )</td>
</tr>
<tr>
<td>( A \alpha^n u[n] )</td>
<td>( K \alpha^n u[n] )</td>
</tr>
<tr>
<td>( A \cos[\omega_n n] + B \sin[\omega_n n] )</td>
<td>( K_1 \cos[\omega_n n] + K_2 \sin[\omega_n n] )</td>
</tr>
</tbody>
</table>

If the characteristic polynomial has a root at the value of an input exponential (e.g., \( \lambda_1 = \frac{1}{2} \) and \( x[n] = \left( \frac{1}{2} \right)^n \)) you would use \( K n \lambda_k^n \) for \( y_p[n] \) (e.g., \( y_p[n] = K n \left( \frac{1}{2} \right)^n \)).

There are \( M - N \ D_i \)'s. If \( M \leq N \) there are no \( D_i \)'s.

To find the impulse response, there is no \( y_p[n] \), and there are \( M - N + 1 \ D_i \)'s. If \( M < N \) there are no \( D_i \)'s for the impulse response.

Solve for the unknowns by finding \( y[0] \), \( y[1] \), \( y[2] \), \( \cdots \) until you get as many equations as you have unknowns. Solve these equations for the unknowns.

You can solve the difference equation for the case \( x[n] = 0 \), subject to the initial conditions of the system. This is the natural response or zero-input response, \( y_{z1}[n] \), of the system. You can further solve for the case \( y[n] = 0 \) for \( n < 0 \) for the actual input \( x[n] \). This is the forced response or zero-state response, \( y_{z1}[n] \), of the system. The total response is the sum of the two: \( y[n] = y_{z1}[n] + y_{z2}[n] \).

A system is BIBO stable if, for every bounded input, the output is bounded. If the input \( x[n] \) is bounded, then the particular solution \( y_p[n] \) will be bounded. Thus, the only possible unbounded terms in Eq. (1) are the \( \lambda_k^n \)'s. These terms are bounded (\( \lambda_k^n \to 0 \) as \( n \to \infty \)) if \( |\lambda_k| < 1 \). Hence, the system is BIBO stable if \( |\lambda_k| < 1 \) for all the \( k \)'s. (\( \lambda_k^n \) doesn’t blow up if \( |\lambda_k| = 1 \). However, if \( |\lambda_k| = 1 \), the input \( x[n] = \lambda_k^n \) will produce the output \( y[n] = C_1 \lambda_k^n + K n \lambda_k^n \), and the \( K n \lambda_k^n \) term will blow up as \( n \to \infty \).)